Axiomatic classes of models in modal logics

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To worm up...

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Theorem (Keisler, 1961)

Let $\mathbb{K}$ be any class of first-order models (of a fixed signature).

- $\mathbb{K}$ *is axiomatizable* $\iff$
  $\mathbb{K}$ *is closed under* $\equiv_{\mathsf{FO}}$ *and ultraproducts*

Elementary equivalence: $M \equiv_{\mathsf{FO}} N$ means:

for every formula $A \in \mathsf{FO}$ we have $\big(M \models A \iff N \models A \big)$

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	Closure conditions		
	Both $\mathbb{K}$ and $\overline{\mathbb{K}}$	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K}$ is axiomatizable	$\equiv_{\mathsf{FO}}$	uProd	
$\mathbb{K}$ is finitely axiomatizable	$\equiv_{\mathsf{FO}}$	uProd	uProd

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Why non-symmetric?

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Why non-symmetric?
Because this is not the whole story!

The 4 "species" of classes

Definition. For a class of models $\mathbb{K}$ we write:

$\mathbb{K} \in \mathbb{L}$ if $\mathbb{K} = \mathsf{Models}(A)$, for some formula $A \in \mathsf{FO}$.

The 4 "species" of classes

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$\mathbb{K} \in \bigcap\mathbb{L}$ if $\mathbb{K} = \mathrm{Models}(\Gamma)$, for some set of formulas $\Gamma \subseteq \mathrm{FO}$.

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Equivalently: if $\mathbb{K} = \bigcap_{i \in I} \mathbb{K}_i$ for some classes $\mathbb{K}_i \in \mathbb{L}$.

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$\mathbb{K} \in \mathbb{L}$	if	$\mathbb{K} = \mathsf{Models}(A)$, for some formula $A \in \mathsf{FO}$.
$\mathbb{K} \in \capdot\mathbb{L}$	if	$\mathbb{K} = \mathsf{Models}(\Gamma)$, for some set of formulas $\Gamma \subseteq \mathsf{FO}$.
Equivalently:	if	$\mathbb{K} = \bigcap_{i \in I} \mathbb{K}_i$ for some classes $\mathbb{K}_i \in \mathbb{L}$.
$\mathbb{K} \in \cupdot\mathbb{L}$	if	$\mathbb{K} = \bigcup_{i \in I} \mathbb{K}_i$ for some classes $\mathbb{K}_i \in \mathbb{L}$.

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$\mathbb{K} \in \dot{\cup}\mathbb{L}$	if	$\mathbb{K} = \bigcup_{i \in I} \mathbb{K}_i$ for some classes $\mathbb{K}_i \in \mathbb{L}$.
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	if	$\mathbb{K} = \bigcup_{i \in I} \bigcap_{j \in J_i} \mathbb{K}_{i,j}$ for some classes $\mathbb{K}_{i,j} \in \mathbb{L}$.

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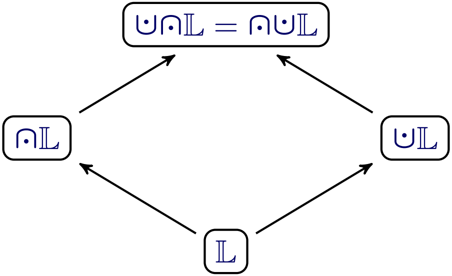
A somewhat “oldish” terminology:

$\mathbb{K} \in \mathbb{L}$	— an	elementary class of models (*finitely axiomatizable*)
$\mathbb{K} \in \capdot\mathbb{L}$	— a	Δ-elementary class of models (*axiomatizable*)
$\mathbb{K} \in \cupdot\mathbb{L}$	— a	Σ-elementary class of models (*co-axiomatizable?*)
$\mathbb{K} \in \cupdot\capdot\mathbb{L}$	— a	$\Sigma\Delta$-elementary class of models

The hierarchy of the 4 species of classes



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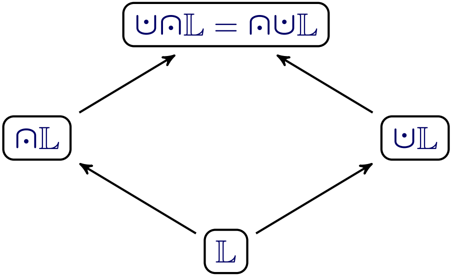

- In $\mathbb{L}$: the classes of all groups (rings, fields)
- In $\dot{\cap}\mathbb{L}$: infinite groups (rings, fields), algebraically closed fields
- In $\dot{\cup}\mathbb{L}$: finite groups (rings, fields)
- In $\dot{\cup}\dot{\cap}\mathbb{L}$: infinite fields of characteristic $p > 0$;
  infinite finitely dimensional vector spaces
- Not even in $\dot{\cup}\dot{\cap}\mathbb{L}$: well-ordered sets, periodic groups, simple groups

Theorem (Keisler, 1961)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{FO}}$		
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\equiv_{\mathsf{FO}}$		uProd
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{FO}}$	uProd	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{FO}}$	uProd	uProd

Legend: uProd = ultraproduct

Theorem (Keisler, 1961) **(Keisler, 1961; Shelah, 1971)**

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$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\equiv_{FO}$		
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\equiv_{FO}$		uProd
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\equiv_{FO}$	uProd	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{FO}$	uProd	uProd

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\cong$	uPow	uPow
$\cong$	uPow	uProd
$\cong$	uProd	uPow
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$\mathbb{K} \in \dot\cap\mathbb{L}$	$\equiv_{FO}$	uProd	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{FO}$	uProd	uProd

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$\cong$	uPow	uPow
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$\cong$	uProd	uPow
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Main reason for the symmetry in the tables:

$$M \not\models A \iff M \models \neg A$$

Formulas: $p_i \;|\; \neg A \;|\; (A \wedge B) \;|\; (A \vee B) \;|\; (A \to B) \;|\; \Box A$

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Kripke semantics:

Kripke model: $M = (W, R, V)$, where

- $W \neq \varnothing$ — a nonempty set of worlds
- $R \subseteq W \times W$ — a accessibility relation between worlds
- $V(p_i) \subseteq W$ — a valuation of variables

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Truth of a formula is defined in a pointed model (M, x):

$$
\begin{array}{lcl}
M, x \models p_i & \leftrightharpoons & x \in V(p_i) \\
M, x \models \neg A & \leftrightharpoons & M, x \not\models A \\
M, x \models A \wedge B & \leftrightharpoons & M, x \models A \ \text{ and } \ M, x \models B \\
M, x \models A \vee B & \leftrightharpoons & M, x \models A \ \text{ or } \ M, x \models B \\
M, x \models A \to B & \leftrightharpoons & M, x \models A \ \Rightarrow \ M, x \models B \\
M, x \models \Box A & \leftrightharpoons & \text{for every } y \in W \ (x R y \ \Rightarrow \ M, y \models A)
\end{array}
$$

Truth of a formula in a model: $M \models A \ \leftrightharpoons \ \forall x \in W \ \ M, x \models A.$

Modal equivalence of two (pointed) Kripke models

$M \equiv_{\mathsf{ML}} N \quad \leftrightharpoons \quad$ for every formula $A \in \mathsf{ML}$: $\quad M \models A \iff N \models A$

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Bisimulation between two pointed Kripke models

$M, a \simeq N, b$ —
- respects the valuation of variables
- every step in M is "simulated" by some step in N
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Modal language | Relations between (pointed) models

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Global bisimulation between Kripke models

$M :\simeq: N$ — bisimulation that covers the whole models M and N

Compact and saturated classes of structures

Let $\mathbb{K}$ be a class of pointed Kripke models: (M, a)

Definition

$\mathbb{K}$ is called (modally) compact if, for every set of modal formulas $\Gamma \subseteq \mathrm{ML}$,

$$\text{every finite subset } \Delta \subseteq \Gamma \text{ is satisfiable in the class } \mathbb{K} \\ \Longrightarrow \quad \Gamma \text{ is satisfiable in the class } \mathbb{K}$$

Compact and saturated classes of structures

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Definition

$\mathbb{K}$ is called (modally) saturated if

for every $(M, a) \in \mathbb{K}$ there is $(M^\sharp, a^\sharp) \in \mathbb{K}$ such that:
1) $(M, a) \equiv_{\mathsf{ML}} (M^\sharp, a^\sharp)$;
2) the model $M^\sharp$ is modally saturated.

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Fact 1.

For arbitrary models M, N: $(M, a) \simeq (N, b) \implies (M, a) \equiv_{\mathsf{ML}} (N, b)$

Fact 2.

For saturated models M, N: $(M, a) \simeq (N, b) \iff (M, a) \equiv_{\mathsf{ML}} (N, b)$

Definability: classes of pointed models: abstract formulation

Theorem (Z, 2017)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML}}$		
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\equiv_{\mathsf{ML}}$		compact
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML}}$	compact	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML}}$	compact	compact

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Can we replace the "linguistic" relation of modal equivalence $\equiv_{\mathsf{ML}}$ with the "structural" relation of bisimulation $\simeq$?
Yes, at the price of adding saturatedness of $\mathbb{K}$ and $\overline{\mathbb{K}}$.

Definability: classes of pointed models: abstract formulation

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	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cupdot\Capdot\mathbb{L}$	$\equiv_{\mathsf{ML}}$		
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$\mathbb{K} \in \Capdot\mathbb{L}$	$\equiv_{\mathsf{ML}}$	compact	
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Theorem (Z, 2017)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cupdot\Capdot\mathbb{L}$	$\simeq$	saturated	saturated
$\mathbb{K} \in \Cupdot\mathbb{L}$	$\simeq$	saturated	compact saturated
$\mathbb{K} \in \Capdot\mathbb{L}$	$\simeq$	compact saturated	saturated
$\mathbb{K} \in \mathbb{L}$	$\simeq$	compact saturated	compact saturated

Definability: classes of pointed models: concrete results

Theorem (Basic ML; de Rijke, 1993)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \cupdot\capdot\mathbb{L}$	$\equiv_{\mathsf{ML}}$		
$\mathbb{K} \in \cupdot\mathbb{L}$	$\equiv_{\mathsf{ML}}$		uProd
$\mathbb{K} \in \capdot\mathbb{L}$	$\equiv_{\mathsf{ML}}$	uProd	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML}}$	uProd	uProd

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$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML}}$	uProd	uProd

(Tense language)

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\equiv_{\mathsf{ML.t}}$		
$\equiv_{\mathsf{ML.t}}$		uProd
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$\equiv_{\mathsf{ML.t}}$		uProd
$\equiv_{\mathsf{ML.t}}$	uProd	
$\equiv_{\mathsf{ML.t}}$	uProd	uProd

(ML with universal modality)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML}\forall}$		
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\equiv_{\mathsf{ML}\forall}$		uProd
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML}\forall}$	uProd	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML}\forall}$	uProd	uProd

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$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\equiv_{\mathsf{ML}\forall}$		uProd
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(Graded ML, de Rijke, 2000)

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\equiv_{\mathsf{MLG}}$		
$\equiv_{\mathsf{MLG}}$		uProd
$\equiv_{\mathsf{MLG}}$	uProd	
$\equiv_{\mathsf{MLG}}$	uProd	uProd

Definability: classes of pointed models: concrete

Theorem (Basic ML; de Rijke, 1993)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot\cup\dot\cap\mathbb{L}$	$\simeq$	uPow	uPow
$\mathbb{K} \in \dot\cup\mathbb{L}$	$\simeq$	uPow	uProd
$\mathbb{K} \in \dot\cap\mathbb{L}$	$\simeq$	uProd	uPow
$\mathbb{K} \in \mathbb{L}$	$\simeq$	uProd	uProd

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Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
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$\simeq_t$	uProd	uPow
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(ML with universal modality)

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$\mathbb{K} \in \dot\cup\dot\cap\mathbb{L}$	$:\simeq:$	uPow	uPow
$\mathbb{K} \in \dot\cup\mathbb{L}$	$:\simeq:$	uPow	uProd
$\mathbb{K} \in \dot\cap\mathbb{L}$	$:\simeq:$	uProd	uPow
$\mathbb{K} \in \mathbb{L}$	$:\simeq:$	uProd	uProd

(Graded ML, de Rijke, 2000)

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\simeq_G$	uPow	uPow
$\simeq_G$	uPow	uProd
$\simeq_G$	uProd	uPow
$\simeq_G$	uProd	uProd

Modal language: “purely modal” operations on models

Ultra-extension of a Kripke model $M = (W, R, V)$

is a Kripke model $M^{\mathfrak{ue}} = (W^{\mathfrak{ue}}, R^{\mathfrak{ue}}, V^{\mathfrak{ue}})$, where

worlds:	$W^{\mathfrak{ue}}$	—	all ultrafilters over the set W;
relation:	$\alpha\, R^{\mathfrak{ue}}\, \beta$	$\leftrightharpoons$	$\forall X \subseteq W \; (\Diamond X \in \alpha \;\Leftarrow\; X \in \beta)$
		$\Leftrightarrow$	$\forall X \subseteq W \; (\Box X \in \alpha \;\Rightarrow\; X \in \beta)$
valuation:	$\alpha \models p_i$	$\leftrightharpoons$	$V(p_i) \in \alpha$

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Observation. 1) $M \equiv_{\mathrm{ML}} M^{\mathfrak{ue}}$; 2) the model $M^{\mathfrak{ue}}$ is modally saturated.

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		$\Leftrightarrow$	$\forall X \subseteq W \; (\Box X \in \alpha \;\Rightarrow\; X \in \beta)$
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Observation. 1) $M \equiv_{\mathrm{ML}} M^{\mathfrak{ue}}$; 2) the model $M^{\mathfrak{ue}}$ is modally saturated.

Ultra-union of a family of pointed Kripke models $(M_i, a_i)_{i \in I}$

$M = \Big(\big(\biguplus_{i \in I} M_i \big)^{\mathfrak{ue}}, \; \alpha \Big)$, all co-finite subsets of $\{ \langle a_i, i \rangle \mid i \in I \}$ are in α.

Observation. Ultra-union behaves like the ultra-product.

Definability: classes of pointed models: "pure" criteria

Theorem (Yde Venema, 1999)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot\cup\dot\cap\mathbb{L}$	$\equiv_{ML}$		
$\mathbb{K} \in \dot\cup\mathbb{L}$	$\equiv_{ML}$		$\uplus^{ue}$
$\mathbb{K} \in \dot\cap\mathbb{L}$	$\equiv_{ML}$	$\uplus^{ue}$	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{ML}$	$\uplus^{ue}$	$\uplus^{ue}$

Definability: classes of pointed models: "pure" criteria

Theorem (Yde Venema, 1999)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\simeq$	$\mathfrak{ue}$	$\mathfrak{ue}$
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\simeq$	$\mathfrak{ue}$	$\uplus^{\mathfrak{ue}}$
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\simeq$	$\uplus^{\mathfrak{ue}}$	$\mathfrak{ue}$
$\mathbb{K} \in \mathbb{L}$	$\simeq$	$\uplus^{\mathfrak{ue}}$	$\uplus^{\mathfrak{ue}}$

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$\mathbb{K} \in \mathbb{L}$	$\simeq$	$\uplus^{\mathfrak{ue}}$	$\uplus^{\mathfrak{ue}}$

Theorem (Tense language) (exercise)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML.t}}$		
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\equiv_{\mathsf{ML.t}}$		$\uplus^{\mathfrak{ue}}$
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML.t}}$	$\uplus^{\mathfrak{ue}}$	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML.t}}$	$\uplus^{\mathfrak{ue}}$	$\uplus^{\mathfrak{ue}}$

Definability: classes of pointed models: "pure" criteria

Theorem (Yde Venema, 1999)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\simeq$	$\mathfrak{ue}$	$\mathfrak{ue}$
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\simeq$	$\mathfrak{ue}$	$\uplus^{\mathfrak{ue}}$
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\simeq$	$\uplus^{\mathfrak{ue}}$	$\mathfrak{ue}$
$\mathbb{K} \in \mathbb{L}$	$\simeq$	$\uplus^{\mathfrak{ue}}$	$\uplus^{\mathfrak{ue}}$

Theorem (Tense language) (exercise)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\simeq_t$	$\mathfrak{ue}$	$\mathfrak{ue}$
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\simeq_t$	$\mathfrak{ue}$	$\uplus^{\mathfrak{ue}}$
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\simeq_t$	$\uplus^{\mathfrak{ue}}$	$\mathfrak{ue}$
$\mathbb{K} \in \mathbb{L}$	$\simeq_t$	$\uplus^{\mathfrak{ue}}$	$\uplus^{\mathfrak{ue}}$

Now what about classes of models?

Theorem (Lines 1 and 3: Z, 2017; Lines 2 and 4: open questions)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$	
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML}}$	$\hookrightarrow$	$\uplus$ compact	✓
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\equiv_{\mathsf{ML}}$	$\hookrightarrow$	$\uplus$ compact	?
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML}}$	$\hookrightarrow$ $\uplus$ compact	$\uplus$ compact	✓
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML}}$	$\hookrightarrow$ $\uplus$ compact	$\uplus$ compact	?

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$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\equiv_{\mathsf{ML}}$	$\hookrightarrow$	$\uplus$ compact	?
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML}}$	$\hookrightarrow$ $\uplus$ compact	$\uplus$ compact	✓
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML}}$	$\hookrightarrow$ $\uplus$ compact	$\uplus$ compact	?

Now we get rid of $\equiv_{\mathsf{ML}}$ in favour of bisimulation $\simeq$.

Now what about classes of models?

Theorem (Lines 1 and 3: Z, 2017; Lines 2 and 4: open questions)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$	
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\equiv_{\mathrm{ML}}$	$\hookrightarrow$	$\uplus$ compact	✓
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\equiv_{\mathrm{ML}}$	$\hookrightarrow$	$\uplus$ compact	?
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\equiv_{\mathrm{ML}}$	$\hookrightarrow$ $\uplus$ compact	$\uplus$ compact	✓
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathrm{ML}}$	$\hookrightarrow$ $\uplus$ compact	$\uplus$ compact	?

Theorem (Lines 1 and 3: Z, 2017; Lines 2 and 4: open questions)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$	
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$:\simeq:$	$\hookrightarrow$ saturated	$\uplus$ compact saturated	✓
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$:\simeq:$	$\hookrightarrow$ saturated	$\uplus$ compact saturated	?
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$:\simeq:$	$\hookrightarrow$ $\uplus$ compact saturated	$\uplus$ compact saturated	✓
$\mathbb{K} \in \mathbb{L}$	$:\simeq:$	$\hookrightarrow$ $\uplus$ compact saturated	$\uplus$ compact saturated	?

Definability: classes of models: basic modal language

Theorem (Lines 3-4: de Rijke, Sturm, 2001; Lines 1–2: Z, 2017)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML}}$	$\hookrightarrow$	
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\equiv_{\mathsf{ML}}$	$\hookrightarrow$	uProd
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML}}$	$\hookrightarrow$ $\uplus$ uProd	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML}}$	$\hookrightarrow$ $\uplus$ uProd	uProd

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$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\equiv_{ML}$	$\hookrightarrow$	
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\equiv_{ML}$	$\hookrightarrow$	uProd
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\equiv_{ML}$	$\hookrightarrow$ $\uplus$ uProd	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{ML}$	$\hookrightarrow$ $\uplus$ uProd	uProd

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$:\simeq:$	$\hookrightarrow$ uPow	uPow
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$:\simeq:$	$\hookrightarrow$ uPow	uProd
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$:\simeq:$	$\hookrightarrow$ $\uplus$ uProd	uPow
$\mathbb{K} \in \mathbb{L}$	$:\simeq:$	$\hookrightarrow$ $\uplus$ uProd	uProd

Definability: classes of models: "pure" criteria

Theorem (Line 3: Yde Venema, 1999; Line 1: Z, 2017)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\equiv_{ML}$	$\hookrightarrow$	
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	?		
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\equiv_{ML}$	$\hookrightarrow$ $\uplus$ $\mathfrak{ue}$	
$\mathbb{K} \in \mathbb{L}$	?		

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$:\simeq:$	$\hookrightarrow$ $\mathfrak{ue}$	$\mathfrak{ue}$
?		
$:\simeq:$	$\hookrightarrow$ $\uplus$ $\mathfrak{ue}$	$\mathfrak{ue}$
?		

Definability: classes of models: ML with universal modality

We can "internalize" negation:

$$M \not\models \varphi \quad \Longleftrightarrow \quad M \models \neg[\forall]\varphi.$$

Hence, the symmetry is again with us!

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Theorem (ML with universal modality; Z, 2017)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \cupdot\capdot\mathbb{L}$	$\equiv_{\mathsf{ML}\forall}$		
$\mathbb{K} \in \cupdot\mathbb{L}$	$\equiv_{\mathsf{ML}\forall}$		uProd
$\mathbb{K} \in \capdot\mathbb{L}$	$\equiv_{\mathsf{ML}\forall}$	uProd	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML}\forall}$	uProd	uProd

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$:\simeq:$	uPow	uPow
$:\simeq:$	uPow	uProd
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	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
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$\mathbb{K} \in \cupdot\mathbb{L}$	$\equiv_{ML\forall}$		uProd
$\mathbb{K} \in \capdot\mathbb{L}$	$\equiv_{ML\forall}$	uProd	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{ML\forall}$	uProd	uProd

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$:\simeq:$	uPow	uPow
$:\simeq:$	uPow	uProd
$:\simeq:$	uProd	uPow
$:\simeq:$	uProd	uProd

Question

What are "purely modal" criteria for the ML with universal modality?

Further directions

- Criteria for other semantics of the basic modal language:
  - neighbourhood semantics
  - topological semantics
  - algebraic semantics

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    - $\mathbb{L}$: classes of models definable by a single formula
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- Will the results survive if we go to finite models?

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Thank you!