Kripke completeness of strictly positive modal logics

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Joint work with **Stanislav Kikot, Agi Kurucz, Yoshihito Tanaka** and **Frank Wolter**

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Based on the (submitted) paper

S. Kikot, A. Kurucz, Y. Tanaka, F. Wolter & M. Zakharyaschev

Kripke Completeness of Strictly Positive Modal Logics over Meet-semilattices with Operators

https://arxiv.org/abs/1708.03403

$\mathcal{SP}$-terms, equations, and theories

Strictly positive terms (or $\mathcal{SP}$***-terms***) are defined by the grammar

$$\sigma \quad ::= \quad p_i \quad | \quad \top \quad | \quad \Diamond_j \sigma \quad | \quad \sigma \wedge \sigma'$$

where p_i are propositional variables

$\mathcal{SP}$-equation takes the form $e = (\sigma \leq \tau)$ $\quad \sigma, \tau$ are $\mathcal{SP}$-terms

($\mathcal{SP}$-implication)

($\mathcal{SP}$-sequent)

NB $\mathcal{SP}$-equations are always **Sahlqvist formulas** in Modal Logic

$\mathcal{SP}$-theory (or **logic**) is a set, $\mathcal{E}$, of $\mathcal{SP}$-equations

$\mathcal{SP}$-terms, equations, and theories

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Edith Hemaspaandra (2001) called terms with $p_i, \neg p_i, \wedge, \Diamond_i, \Box_i$

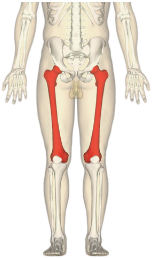
poor man's formulas

Who needs the **pauper's** $\mathcal{SP}$-terms, equations, and theories?

$\mathcal{SP}$-theories in Knowledge Representation

Description logic $\mathcal{EL}$ and the **OWL 2 EL** profile of
the Web Ontology Language OWL 2

SNOMED CT



$$
\begin{aligned}
\text{EntireFemur} &\sqsubseteq \text{StructureOfFemur}\\
\text{FemurPart} &\sqsubseteq \text{StructureOfFemur} \sqcap \exists\text{partOf}.\text{EntireFemur}\\
\text{BoneStructureOfDistalFemur} &\sqsubseteq \text{FemurPart}\\
\text{EntireDistalFemur} &\sqsubseteq \text{BoneStructureOfDistalFemur}\\
\text{DistalFemurPart} &\sqsubseteq \text{BoneStructureOfDistalFemur} \sqcap \exists\text{partOf}.\text{EntireDistalFemur}
\end{aligned}
$$

Comprehensive healthcare terminology with approximately 400 000 definitions
(400 000 concept names and 60 binary relations)

OWL 2 is **undecidable**, OWL 2 DL ($\mathcal{SROIQ}$) is **2NExpTime-complete**

$\mathcal{EL}$ is tractable validity of quasi-equations w.r.t. first-order semantics, i.e., **Kripke models**

consequence relation 1: $\mathcal{E} \models_{\mathsf{Kr}} e$

Descri

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Compre ns

ations)

OWL 2 is

odels

$\mathcal{SP}$-definable first-order properties

first-order property	$\mathcal{SP}$-equations	notation
reflexivity	$p \leq \Diamond p$	e_{refl}
transitivity	$\Diamond\Diamond p \leq \Diamond p$	e_{trans}
symmetry	$q \wedge \Diamond p \leq \Diamond(p \wedge \Diamond q)$	e_{sym}
$\forall x, y, z\,\big(R(x,y) \wedge R(x,z) \to R(y,z)\big)$ Euclideanness	$\Diamond p \wedge \Diamond q \leq \Diamond(p \wedge \Diamond q)$	e_{eucl}
quasi-order	$\{e_{\mathit{refl}}, e_{\mathit{trans}}\}$	$\mathcal{E}_{\mathbf{S4}}$
equivalence	$\{e_{\mathit{refl}}, e_{\mathit{trans}}, e_{\mathit{sym}}\}$ $\{e_{\mathit{refl}}, e_{\mathit{trans}}, e_{\mathit{eucl}}\}$	$\mathcal{E}_{\mathbf{S5}}$ $\mathcal{E}'_{\mathbf{S5}}$
$\forall x, y, z\,\big[R(x,y) \wedge R(x,z) \to$ $\big(R(y,y) \wedge R(y,z)\big) \vee \big(R(z,z) \wedge R(z,y)\big)\big]$	$\Diamond(p \wedge q) \wedge \Diamond(p \wedge r) \leq$ $\Diamond(p \wedge \Diamond q \wedge \Diamond r)$	e_{wcon}
linear quasi-order	$\{e_{\mathit{refl}}, e_{\mathit{trans}}, e_{\mathit{wcon}}\}$	$\mathcal{E}_{\mathbf{S4.3}}$
$\forall x, y\,\big[R(x,y) \to \exists z\,\big(R(x,z) \wedge R(z,y)\big)\big]$ density	$\Diamond p \leq \Diamond\Diamond p$	e_{dense}
$\forall x, y, z\,\big(R(x,y) \wedge R(x,z) \to (y = z)\big)$ functionality	$\Diamond p \wedge \Diamond q \leq \Diamond(p \wedge q)$	e_{fun}

$\mathcal{SP}$-undefinable first-order properties

by a **general necessary condition for $\mathcal{SP}$-definability**

first-order property	modal formula(s)	notation
pseudo-transitivity $\forall x, y, z\, (R(x, y) \wedge R(y, z) \to R(x, z) \vee (x = z))$	$\Diamond\Diamond p \leq p \vee \Diamond p$	φ_{ptrans}
pseudo-equivalence	e_{sym}, φ_{ptrans}	**Diff**
$\forall x, y, z\, (R(x, y) \wedge R(x, z) \to R(y, z) \vee R(z, y) \vee (y = z))$ weak connectedness	$\Diamond p \wedge \Diamond q \leq \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond q)$	φ_{wcon}
transitivity and weak connectedness	e_{trans}, φ_{wcon}	**K4.3**
confluence $\forall x, y, z\, (R(x, y) \wedge R(x, z) \to \exists u\, (R(y, u) \wedge R(z, u)))$	$\Diamond\Box p \leq \Box\Diamond p$	φ_{conf}
transitivity and confluence	e_{trans}, φ_{conf}	**K4.2**
transitivity and $\forall x \exists y\, (R(x, y) \wedge \forall z\, (R(y, z) \to (y = z)))$	e_{trans}, $\Box\Diamond p \leq \Diamond\Box p$	**K4.1**

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first-order property	modal formula(s)	notation
$\forall x, y, z\,\big(R(x,y) \wedge R(y,z) \to$ pseudo-transitivity $R(x,z) \vee (x = z)\big)$	$\Diamond\Diamond p \le p \vee \Diamond p$	φ_{ptrans}
pseudo-equivalence	$e_{sym},\ \varphi_{ptrans}$	**Diff**
$\forall x, y, z\,\big(R(x,y) \wedge R(x,z) \to R(y,z) \vee R(z,y) \vee (y = z)\big)$ weak connectedness	$\Diamond p \wedge \Diamond q \le \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond q)$	φ_{wcon}
transitivity and weak connectedness	$e_{trans},\ \varphi_{wcon}$	**K4.3**
$\forall x, y, z\,\big(R(x,y) \wedge R(x,z) \to$ confluence $\exists u\,(R(y,u) \wedge R(z,u))\big)$	$\Diamond\Box p \le \Box\Diamond p$	φ_{conf}
transitivity and confluence	$e_{trans},\ \varphi_{conf}$	**K4.2**
transitivity and $\forall x \exists y\,(R(x,y) \wedge \forall z\,(R(y,z) \to (y = z)))$	$e_{trans},\ \Box\Diamond p \le \Diamond\Box p$	**K4.1**

For any $\mathcal{E} \supseteq \mathcal{E}_{\mathrm{S4}}$, $\mathsf{Kr}_{\mathcal{E}}$ is closed under subframes ⟶

S4.1-frames and **S4.2**-frames are not $\mathcal{SP}$-definable

But $\{e_{refl},\ e_{trans},\ e_{wcon}\}$ defines **S4.3**-frames

$\mathcal{SP}$-undefinable first-order properties

by a **general necessary condition for $\mathcal{SP}$-definability**

first-order property	modal formula(s)	notation
$\forall x,y,z\,\big(R(x,y)\wedge R(y,z)\to$ pseudo-transitivity $R(x,z)\vee(x=z)\big)$	$\Diamond\Diamond p\le p\vee\Diamond p$	φ_{ptrans}
pseudo-equivalence	$e_{sym},\ \varphi_{ptrans}$	**Diff**
$\forall x,y,z\,\big(R(x,y)\wedge R(x,z)\to R(y,z)\vee R(z,y)\vee(y=z)\big)$ weak connectedness	$\Diamond p\wedge\Diamond q\le\Diamond(p\wedge q)\vee\Diamond(p\wedge\Diamond q)\vee\Diamond(q\wedge\Diamond q)$	φ_{wcon}
transitivity and weak connectedness	$e_{trans},\ \varphi_{wcon}$	**K4.3**
$\forall x,y,z\,\big(R(x,y)\wedge R(x,z)\to$ confluence $\exists u\,(R(y,u)\wedge R(z,u))\big)$	$\Diamond\Box p\le\Box\Diamond p$	Svyatlovsky's talk φ_{conf}
transitivity and confluence	$e_{trans},\ \varphi_{conf}$	**K4.2**
transitivity and $\forall x\exists y\,(R(x,y)\wedge\forall z\,(R(y,z)\to(y=z)))$	$e_{trans},\ \Box\Diamond p\le\Diamond\Box p$	**K4.1**

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$\mathcal{SP}$-theories: algebraic view

Bounded meet-semilattices with normal monotone operators (or **SLOs**)

$\mathfrak{A} = (A, \wedge, \top, \Diamond_i)$ ($\sigma \le \tau$ is a shorthand for $\sigma \wedge \tau = \sigma$)

($\sigma = \tau$ is a shorthand for $\sigma \le \tau$ and $\tau \le \sigma$)

- $p \wedge p = p$
- $p \wedge q = q \wedge p$
- $p \wedge (q \wedge r) = (p \wedge q) \wedge r$
- $p \le \top$
- $\Diamond_i(p \wedge q) \le \Diamond_i q$ (*monotonicity*)

$\mathcal{SP}$-theories: algebraic view

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- $\Diamond_i(p \wedge q) \leq \Diamond_i q$ *(monotonicity)*

Birkhoff's equational calculus

$\mathcal{E} \vdash_{\mathsf{SLO}} e$

$\varphi = \varphi$

$\varphi = \psi / \psi = \varphi$

$\varphi = \psi,\ \psi = \chi / \varphi = \chi$

$\varphi = \psi,\ \alpha = \beta / \varphi(\alpha/p) = \psi(\beta/p)$

consequence relation 2: $\mathcal{E} \models_{\mathsf{SLO}} e \iff \mathcal{E} \vdash_{\mathsf{SLO}} e$

$\forall \mathfrak{A}\ (\mathfrak{A} \models \mathcal{E} \implies \mathfrak{A} \models e)$

$\mathcal{SP}$-theories in Provability Logic

Reflection Calculus RC

(Beklemishev 2012, Dashkov 2012)

Birkhoff's equational calculus for SLOs

$+$

$$\Diamond_n\Diamond_n\sigma \leq \Diamond_n\sigma, \quad \Diamond_n\sigma \leq \Diamond_m\sigma, \quad \Diamond_n\sigma \wedge \Diamond_m\sigma \leq \Diamond_n(\sigma \wedge \Diamond_m\sigma) \quad n > m$$

axiomatises the $\mathcal{SP}$-**fragment** of G. Japaridze's provability logic **GLP**

- **RC** is **tractable**, while GLP is **PSpace**-complete
- **RC** is **complete** w.r.t. finite **Kripke frames**
  while GLP is Kripke incomplete
- **RC** preserves main proof-theoretic applications of GLP
- **RC** allows more general arithmetical interpretations

$\mathcal{SP}$-theories in Provability Logic

Reflection Calculus RC

(Beklemishev 201… …v 2012)

Birkhoff's equational calculus for SLOs

$+$

$\Diamond_n \Diamond_n \sigma \leq \Diamond_n \sigma, \quad \Diamond_n \sigma \leq \Diamond_m \sigma, \quad \ldots \Diamond_m \sigma \leq \Diamond_n(\sigma \wedge \Diamond_m \sigma) \quad n > m$

axiomatises the $\mathcal{SP}$-fragme… …aparidze's provability logic **GLP**

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- **RC** i… …**plete** w.r.t. finite **Kripke frames**
  while GLP is Kripke incomplete
- …C preserves main proof-theoretic applications of GLP
- **RC** allows more general arithmetical interpretations

Ditch Boolean modal logics ! Use $\mathcal{SP}$-theories ?

The problem

Kripke completeness: is a given $\mathcal{SP}$-theory $\mathcal{E}$ complete w.r.t. its Kripke frames?

for all $\mathcal{SP}$-equations e, $\quad \mathcal{E} \models_{\mathsf{SLO}} e \quad \overset{?}{\Longleftrightarrow} \quad \mathcal{E} \models_{\mathsf{Kr}} e$

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by Sahlqvist completeness, $\quad \mathcal{E} \models_{\mathsf{Kr}} e \quad \Longleftrightarrow \quad \mathcal{E} \models_{\mathsf{BAO}} e \quad \Longleftrightarrow \quad \mathcal{E} \vdash_{\mathbf{K}} e$

BAO-to-SLO conservativity:

for all $\mathcal{SP}$-equations e, $\quad \mathcal{E} \models_{\mathsf{SLO}} e \quad \overset{?}{\Longleftrightarrow} \quad \mathcal{E} \models_{\mathsf{BAO}} e$

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Axiomatisability: does $\mathcal{E}$ axiomatise the $\mathcal{SP}$-fragment of the Boolean modal logic $\mathcal{L}_{\mathcal{E}} = \mathbf{K} \oplus \mathcal{E}$?

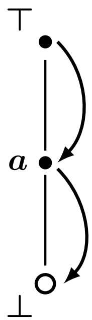
$$\text{for all } \mathcal{SP}\text{-equations } e, \quad \mathcal{E} \models_{\mathsf{SLO}} e \quad \overset{?}{\Longleftrightarrow} \quad e \in \mathcal{L}_{\mathcal{E}}$$

Incomplete $\mathcal{SP}$-theories

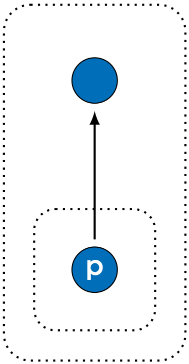
(Kurucz, Tanaka, Wolter & Z, 2010)

$\mathcal{E}_1 = \{\Diamond p \leq p\}$ with FO-correspondent $\forall x, y(R(x,y) \to (x = y))$

Proof: $\mathcal{E}_1 \models_{\mathsf{Kr}} p \wedge \Diamond\top \leq \Diamond p$ but $\mathcal{E}_1 \not\models_{\mathsf{SLO}} p \wedge \Diamond\top \leq \Diamond p$



SLO



'general' frame

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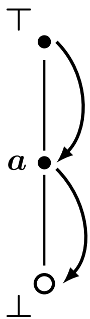
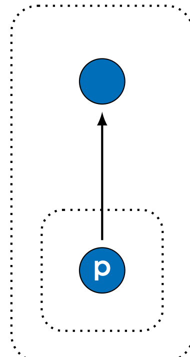


SLO



'general' frame

$\mathcal{E}_2 = \{\Diamond p \leq \Diamond q\}$ with FO-correspondent $R = \emptyset$

in modal logic, Kripke incomplete logics are 'rare' and 'complex'

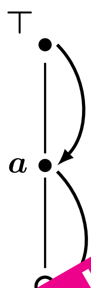
Incomplete $\mathcal{SP}$-theories

(Kurucz, Tanaka, Wolter & Z

$\mathcal{E}_1 = \{\Diamond p \leq p\}$ with FO-correspondent $\forall x, y(R(x, y) \to ($ … $))$

Proof: $\mathcal{E}_1 \models_{\mathsf{Kr}} p \wedge \Diamond\top \leq \Diamond p$ but $\mathcal{E}_1 \not\models_{\mathsf{SLO}} p \wedge \Diamond$ …

Kripke completeness of $\mathcal{SP}$-theories is undecidable



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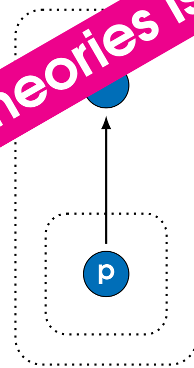

'general' frame

$\mathcal{E}$ … $\{\Diamond p \leq \Diamond q\}$ with FO-correspondent $R = \emptyset$

in modal logic, Kripke incomplete logics are 'rare' and 'complex'

Completeness by canonicity in modal logic

Kripke frame $\mathfrak{F} = (W, R_i)$ ⟶ full complex BAO

$\mathfrak{F}^+ = (2^W, \cup, \cap, -^W, \emptyset, W, \Diamond_i^+)$ $\Diamond_i^+ X = \{w \in W \mid \exists v \in X \; R_i(w, v)\}$

BAO $\mathfrak{A} \models L$ ⟶ $\mathfrak{F}_{\mathfrak{A}} = Uf(\mathfrak{A})$ ⟶ $\mathfrak{F}_{\mathfrak{A}}^+$

$\mathfrak{F}_{\mathfrak{A}}^+ \models L \quad \Longrightarrow \quad L$ is canonical and complete

Can we do something similar for $\mathcal{SP}$-theories and SLOs?

no canonical models

Completeness by complexity

Kripke frame $\mathfrak{F} = (W, R_i)$ ⟶ **SLO-type reduct of full complex BAO**

$\mathfrak{F}^\star = (2^W, \cap, W, \Diamond_i^+)$ $\quad\quad$ $\Diamond_i^+ X = \{w \in W \mid \exists v \in X \; R_i(w, v)\}$

⟶ $\mathcal{E} \models_{\mathsf{SLO}} e \quad \Longrightarrow \quad \mathcal{E} \models_{\mathsf{Kr}} e$

Completeness by complexity

Kripke frame $\mathfrak{F} = (\boldsymbol{W}, \boldsymbol{R}_i)$ ⟶ **SLO-type reduct of full complex BAO**

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⟶ $\mathcal{E} \models_{\mathsf{SLO}} e \quad \Longrightarrow \quad \mathcal{E} \models_{\mathsf{Kr}} e$

An $\mathcal{SP}$-theory $\mathcal{E}$ is **complex** if every SLO $\mathfrak{A} \models \mathcal{E}$ is embeddable into $\mathfrak{F}^\star$
for some Kripke frame $\mathfrak{F} \models \mathcal{E}$

$\mathcal{E}$ **is complex** $\Longrightarrow$ $\mathcal{E}$ **is complete**

Completeness by complexity

Kripke frame $\mathfrak{F} = (W, R_i)$ → **SLO-type reduct of full complex BAO**

$\mathfrak{F}^\star = (2^W, \cap, W, \Diamond_i^+)$

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→ $\mathcal{E} \models_{\mathsf{SLO}} e \quad \Longrightarrow \quad \mathcal{E} \models_{\mathsf{Kr}} e$

An $\mathcal{SP}$-theory $\mathcal{E}$ is **complex** if every SLO $\mathfrak{A} \models \mathcal{E}$ is embeddable into $\mathfrak{F}^\star$ for some Kripke frame $\mathfrak{F} \models \mathcal{E}$

$\mathcal{E}$ is complex $\quad \Longrightarrow \quad$ $\mathcal{E}$ is complete

Theorem Every SLO is embeddable into $\mathfrak{F}^\star$, for some Kripke frame $\mathfrak{F}$

(via elements of SLOs or via filters)

↓

The empty $\mathcal{SP}$-theory is complex, and so complete: $\models_{\mathsf{Kr}} e$ implies $\models_{\mathsf{SLO}} e$, for every $\mathcal{SP}$-equation e

Completeness by complexity

Kripke frame $\mathfrak{F} = (W, R_i)$ → SLO-type reduct of full complex BA…

$\mathfrak{F}^\star = (2^W, \cap, W, \Diamond_i^+)$

$\Diamond_i^+ X = \{w \in W$ … $R_i(w, v)\}$

→ $\mathcal{E} \models_{\mathsf{SLO}} e \implies \mathcal{E} \models$ …

An $\mathcal{SP}$-theory $\mathcal{E}$ is **complex** if every SL… is embeddable into $\mathfrak{F}^\star$
for some Kripke frame $\mathfrak{F} \models \mathcal{E}$

$\mathcal{E}$ is comp… $\implies$ $\mathcal{E}$ is complete

Theorem Every S… …beddable into $\mathfrak{F}^\star$, for some Kripke frame $\mathfrak{F}$

(via elements of SLOs or via filters)

↓

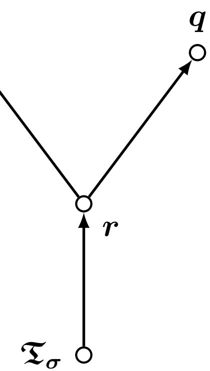
The empty $\mathcal{SP}$-theory is complex, and so complete: $\models_{\mathsf{Kr}} e$ implies $\models_{\mathsf{SLO}} e$,
for every $\mathcal{SP}$-equation e

We have a 'method' but how to classify $\mathcal{SP}$-theories?

Sahlqvist correspondence for $\mathcal{SP}$-equations

$\mathcal{SP}$-terms as **Kripke models**

$$\sigma = \Diamond(r \wedge \Diamond q \wedge \Diamond p) \longrightarrow \mathfrak{M}_\sigma = (W_\sigma, R_\sigma, \mathfrak{v}_\sigma)$$



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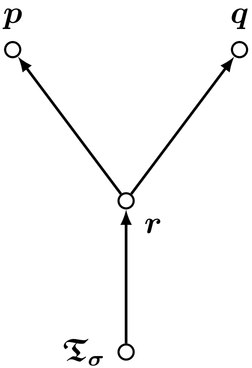
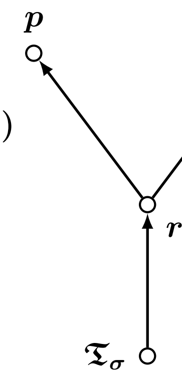


$$\mathfrak{M}, w \models \sigma \iff \exists h\colon \mathfrak{M}_\sigma \to \mathfrak{M} \text{ with } h(r_\sigma) = w$$

$$\models_{\mathsf{Kr}} \sigma \leq \tau \iff \exists h\colon \mathfrak{M}_\tau \to \mathfrak{M}_\sigma \text{ with } h(r_\tau) = r_\sigma$$

Sahlqvist correspondence for $\mathcal{SP}$-equations

$\mathcal{SP}$-terms as **Kripke models**



$$\sigma \;=\; \Diamond(r \wedge \Diamond q \wedge \Diamond p) \;\longrightarrow\; \mathfrak{M}_\sigma = (W_\sigma, R_\sigma, \mathfrak{v}_\sigma)$$

$\mathfrak{M}, w \models \sigma \iff \exists h\colon \mathfrak{M}_\sigma \to \mathfrak{M}$ with $h(r_\sigma) = w$

$\models_{\mathsf{Kr}} \sigma \leq \tau \iff \exists h\colon \mathfrak{M}_\tau \to \mathfrak{M}_\sigma$ with $h(r_\tau) = r_\sigma$

Sahlqvist's correspondence:

every $\mathcal{SP}$-equation $e = (\sigma \leq \tau)$ has the **FO-correspondent**

$$\Psi_e \;=\; \forall \vec{v}_{\mathrm{in}\ \sigma} \Big(\bigwedge_{R_\sigma(v,v')} R(v,v') \;\to\; \exists \vec{u}_{\mathrm{in}\ \tau} \big((r_\sigma = r_\tau) \wedge \bigwedge_{R_\tau(u,u')} R(u,u') \wedge \bigwedge_{u \in \mathfrak{v}_\tau(p)} \bigvee_{v \in \mathfrak{v}_\sigma(p)} (u = v) \big) \Big)$$

for any Kripke frame $\mathfrak{F}$, $\quad \mathfrak{F} \models e \iff \mathfrak{F} \models \Psi_e$

Completeness and FO-correspondents

Systematic approach: investigate completeness of $\mathcal{SP}$-theories based on the form of their FO-correspondents

- universal Horn formulas without $=$ $\quad \forall x, y, z \, (R(x,y) \wedge R(x,z) \to R(y,z))$
- universal Horn formulas with $=$ $\quad \forall x, y, z \, (R(x,y) \wedge R(x,z) \to (y = z))$
- formulas with $\vee$ $\quad \forall x, y, z \, [R(x,y) \wedge R(x,z) \to (R(y,y) \wedge R(y,z)) \vee (R(z,z) \wedge R(z,y))]$
- formulas with $\exists$ $\quad \forall x, y \, [R(x,y) \to \exists z \, (R(x,z) \wedge R(z,y))]$

NB no $\exists$ → closed under **subframes**

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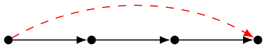
NB no $\exists$ → closed under **subframes**

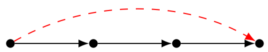
- Every complete subframe $\mathcal{SP}$-theory $\mathcal{E}$ has the polynomial model property, and so is decidable in coNP if $\mathcal{E}$ is finite
- Every complete and finite $\mathcal{SP}$-theory with Horn correspondents is decidable in PTime

$\mathcal{SP}$-equations with Horn correspondents



rooted profile π

$\mathcal{SP}$-equations with Horn correspondents



rooted profile π

'standard' equations

$e_\pi = \Diamond\Diamond\Diamond p \leq \Diamond p$ type 1

$e'_\pi = p_1 \wedge \Diamond(p_2 \wedge \Diamond(p_3 \wedge \Diamond p_4)) \leq p_1 \wedge \Diamond p_4$ type 2

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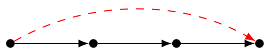

rooted profile π

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e.g., $\Diamond_1 \dots \Diamond_n p \leq \Diamond_0 p$

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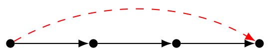

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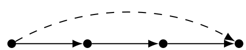
'non-standard' equations

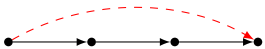
$p \leq \Diamond\Diamond(p \wedge \Diamond p)$ for $\pi =$  $e_\pi = e'_\pi = (p \leq \Diamond p)$

$\Diamond\Diamond p \wedge \Diamond\Diamond\Diamond p \leq \Diamond p$ for 

are incomplete

$\mathcal{SP}$-equations with Horn correspondents

‘standard’ equations



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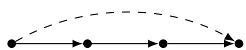
$\Diamond\Diamond p \wedge \Diamond\Diamond\Diamond p \leq \Diamond p$ for 

are incomplete

Normal modal logics axiomatisable by $\mathcal{SP}$-equations can be **undecidable** (Kikot, Shapirovsky, Zolin 2014):

$\Diamond_R \Diamond_P \Diamond_R p \leq \Diamond_P p, \quad \Diamond_Q \Diamond_R p \leq \Diamond_Q p, \quad \Diamond_Q \Diamond_P p \leq \Diamond_P p$

however, the corresponding $\mathcal{SP}$-theory is **tractable**

$\mathcal{SP}$-equations with existential correspondents

Theorem: Any $\mathcal{EL}$-theory $\mathcal{E}$ consisting of equations $e = (\sigma \leq \tau)$ such that

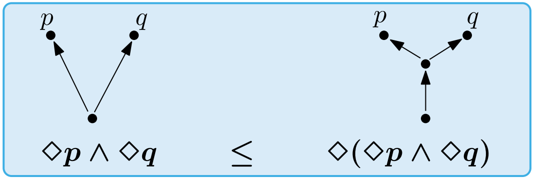
- every variable in σ occurs in it only once,
- τ corresponds to the tree $\mathcal{T}_\tau = (W_\tau, R_\tau, V_\tau)$ with
  - $|W_\tau| \geq 2$ and all points in any $V_\tau(p)$ are leaves of $\mathcal{T}_\tau$,
  - $V_\tau(p) \cap V_\tau(q) = \emptyset$ whenever $p \neq q$

is **complex**, and so **complete**

Example: density axiom $e_{dense} = \Diamond p \leq \Diamond\Diamond p$ with

$$\Psi_{e_{dense}} = \forall x, y \left[R(x, y) \to \exists z \left(R(x, z) \land R(z, y)\right)\right]$$

generalised density



$\mathcal{SP}$-equations with disjunctive correspondents

For $P = \{p_0, \dots, p_n\}$, $n \geq 1$,

$$e^n_{fun} = \Big(\bigwedge_{\substack{Q \subseteq P \\ |Q| = n}} \Diamond(\bigwedge Q) \leq \Diamond(\bigwedge P) \Big)$$

$$\mathcal{E}_{\mathrm{Alt}_n} = \{e^n_{fun}\}$$

$$e^2_{fun} = \big(\Diamond(p \wedge q) \wedge \Diamond(p \wedge r) \wedge \Diamond(q \wedge r) \leq \Diamond(p \wedge q \wedge r)\big)$$

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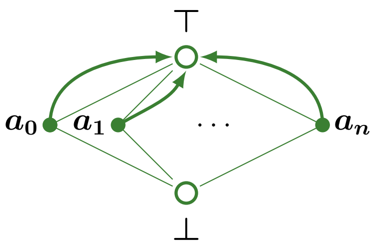
not complex $\mathcal{E}_{\mathrm{Alt}_n}$, $\{e_{refl}, e^n_{fun}\}$, $\{e_{trans}, e^n_{fun}\}$, $\mathcal{E}_{\mathrm{S4}} \cup \{e^n_{fun}\}$, $\mathcal{E}^n_{\mathrm{S5}} = \mathcal{E}_{\mathrm{S5}} \cup \{e^n_{fun}\}$

for $n \geq 2$



$\models \mathcal{E}_{\mathrm{S5}} \cup \{e^n_{fun}\}$

but not embeddable into $\mathfrak{F}^\star$, for any n-functional $\mathfrak{F}$

$\mathcal{SP}$-equations with disjunctive correspondents

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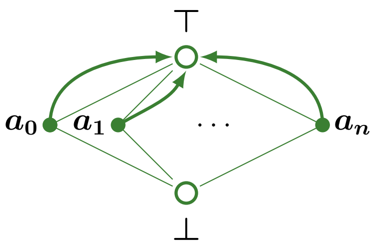

$\models \mathcal{E}_{\mathbf{S5}} \cup \{e^n_{\textit{fun}}\}$

but not embeddable into $\mathfrak{F}^\star$, for any n-functional $\mathfrak{F}$

not complex $\{e_{\textit{wcon}}\}$, $\{e_{\textit{refl}}, e_{\textit{wcon}}\}$, $\mathcal{E}_{\mathbf{S4.3}}$

Completeness by syntactic proxies

$\mathcal{E}$ **is complete** if, for any $e = (\boldsymbol{\sigma} \leq \boldsymbol{\tau})$, $\mathcal{E} \models_{\mathsf{Kr}} e \quad \Longrightarrow \quad \mathcal{E} \models_{\mathsf{SLO}} e$

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(1) $\mathcal{E}$-normal form

$$\mathcal{E} \vdash_{\mathsf{SLO}} \Big(\tau = \bigwedge_{\varrho \in N_\tau} \varrho\Big)$$

reflecting Kripke frames for $\mathcal{E}$

$(\sigma \le \bigwedge_{\varrho \in N_\tau} \varrho)$ is the **syntactic proxy** of e

(2) for any $\varrho \in N_\tau$,

$$\mathcal{E} \models_{\mathsf{Kr}} \sigma \le \varrho \quad \Longrightarrow \quad \mathcal{E}^- \models_{\mathsf{Kr}} \sigma \le \varrho$$

for some **complete** $\mathcal{E}^- \subseteq \mathcal{E}$

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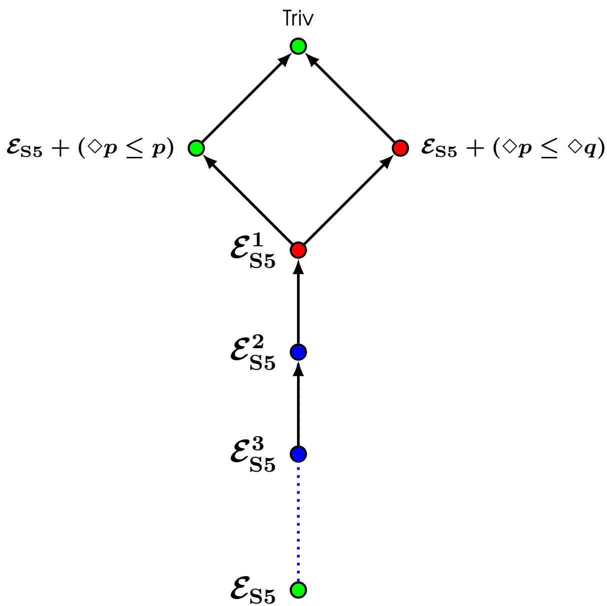
Complete but not complex

– $\mathcal{E}_{\mathsf{Alt}_n}$ $\quad N_\tau = \{\leq n\text{-functional full subtree of } \mathfrak{T}_\tau\}$ $\quad \mathcal{E}^- = \emptyset$

– $\mathcal{E}_{\mathsf{S4.3}}$ $\quad N_\tau = \{\text{full branches of } \mathfrak{T}_\tau\}$ $\quad \mathcal{E}^- = \mathcal{E}_{\mathsf{S4}}$
tractable

Extensions of $\mathcal{E}_{\mathrm{S5}}$

(M. Jackson 2004)



- complex (and so complete)
- complete but not complex
- incomplete