What is Gödel's Second Incompleteness Theorem ???

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De Docta Ignorantia

Most mathematicians know some formulation of Gödel's Second Incompleteness Theorem, G2, but they would be hard-pressed to give a precise statement.

A sufficiently strong theory cannot prove its own consistency.

And what is that supposed to mean? What is *a theory*? What is *sufficiently strong*? What is *a consistency statement*?

G2 has an additional problem: there is no statement of G2 that reflects our intuitive understanding of the theorem.

Mathematicians may be forgiven that they do not see their way around that. Nobody does.

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Ingredient: Theories

What is a theory? Let us fix that a theory is a recursively enumerable theory of finite signature in first-order predicate logic.

There are versions of G2 for theories that are not recursively enumerable, for theories that are not in finite signature, for theories that are not first-order. We will refrain from studying these.

The aim of this talk is not to formulate a most general version.

My other talk at the Wormshop is precisely about the business of G2 in greater generality.

Ingredient: Interpretations

We want our formulation of the theorem to work for set theory, Euclidean geometry, number theory. For that we need the notion of interpretation.

We can interpret arithmetic in set theory, hyperbolic plane geometry in Euclidean plane geometry (and vice versa), etcetera.

An interpretation of V in U is implemented by a translation of the language of V in the language of U. Such a translation sends V-predicates to U-formulas, commutes with the propositional connectives and the quantifiers. We allow some further flexibility: relativization to a domain and more-dimensionality.

- We write $U \rhd V$ for U interprets V.
- We write $U \equiv V$ for: U and V are mutually interpretable, or, i.o.w., $U \rhd V$ and $V \rhd U$.

Ingredient: Basic Theory 1

To formulate our version of G2, we need a weak basic arithmetic B. An amazing discovery is that all reasonable choices for such a theory are the same modulo mutual interpretability.

These theories are also mutually interpretable with equally basic theories of strings, of trees and of sets.

The theory B can be PA^-, the theory of discretely ordered commutative semi-rings with least element.

Alternatively, we can work with a somewhat stronger theory S^1_2 that we will not describe in this talk.

Ingredient: Basic Theory 2

PA^- has the advantage that it very simple and could fool people to think that it is an honest mathematical theory. However, it is not a comfortable theory to reason in.

The theory S^1_2 has the advantage that, for simple axiom sets, we can carry out what is essentially Gödel's reasoning without extra tricks and work-arounds.

The main thing to take home is that these theories are very weak.

Fedor Pakhomov is studying the question whether we can go below our very weak theories.

Ingredient: Basic Theory 3

This is Emil Jeřábek's axiomatization of PA^{-}.

P1. $x + 0 = x$

P2. $x + y = y + x$

P3. $x + (y + z) = (x + y) + z$

P4. $x \cdot 1 = x$

P5. $x \cdot y = y \cdot x$

P6. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

P7. $x \cdot (y + z) = x \cdot y + x \cdot z$

P8. $x \leq y \ \vee \ y \leq x$

P9. $(x \leq y \wedge y \leq z) \rightarrow x \leq z$

P10. $(x + 1) \not\leq x$

P11. $x \leq y \rightarrow (x = y \vee (x + 1) \leq y)$

P12. $x \leq y \rightarrow (x + z) \leq (y + z)$

P13. $x \leq y \rightarrow (x \cdot z) \leq (y \cdot z)$

Ingredient: Formula Class

A formula S is Σ^0_1 if it has the form $\exists \vec{x}\, S_0(\vec{x}, \vec{y})$, where all quantifiers in S_0 are bounded, i.e. of the form $\forall v < t$ and $\exists v < t$.

The Σ^0_1-formulas represent precisely the recursively enumerable sets.

The Π^0_1-formulas are similarly defined, only now we have a block of universal quantifiers.

Ingredient: Arithmetization

We can code all relevant aspects of our proof-system in arithmetic. This gives us an arithmetical predicate $\mathrm{proof}_\alpha(p, a)$ meaning p is the code of a proof from axioms α of the formula coded by a.

We write $\Box_\alpha A$ for $\exists p\ \mathrm{proof}_\alpha(p, \underline{\ulcorner A \urcorner})$. Here $\underline{\ulcorner A \urcorner}$ is the numeral of the Gödel number of A.

We write $\Diamond_\alpha A$ for $\neg \Box_\alpha \neg A$. Thus, $\Diamond_\alpha A$ means: the theory axiomatized by $\alpha + A$ is consistent.

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Statement

G2:
Suppose σ is a Σ^0_1-formula that represents the axiom set of a consistent theory U, then $U \not\rhd (\mathsf{B} + \Diamond_\sigma \top)$.

Note that we eliminated the business of ‘sufficiently strong’ altogether from the statement of the theorem.

Feferman’s Theorem:
Suppose $K : U \rhd \mathsf{B}$, then $U \rhd (U + \Box^K_\sigma \bot)$.

Feferman’s Theorem: a theory can imagine itself to be inconsistent.

EXERCISE: Prove G2 from Feferman’s Theorem.

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Feferman's Axiomatization

Peano Arithmetic (or: PA) is PA^- plus full induction. Let α be a predicate that represents the axioms of PA in a natural way.

Feferman constructs a Π^0_1-predicate $\alpha^\star$ such that $\alpha^\star$ represents the axioms of PA and for all arithmetical sentences A, we have $\mathsf{PA} \vdash \alpha^\star(\ulcorner A \urcorner) \leftrightarrow \alpha(\ulcorner A \urcorner)$. However, we have $\mathsf{PA} \vdash \Diamond_{\alpha^\star} \top$.

Note that it follows that $\mathsf{PA} \nvdash \forall x\, (\alpha^\star(x) \leftrightarrow \alpha(x))$.

Feferman's observation is one of many illustrations that G2 depends on the way the axioms are represented. This phenomenon is called intentionality. Similarly, it may depend on the proof-system. We do not generally have G2 for cut-free systems.

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Coordinates: The Problem

To construct the consistency statement $\Diamond_\alpha \top$, we have to choose, on the meta-level, a treatment of syntax and a proof system. On the object-level, we have to choose a Gödel numbering and ways of presenting the proofs. Thus, there are many conventional choices going in the construction of the sentence.

Feferman's solution to the problem was simply to fix one set of such choices. However, ironically, I do think I understand the theorem, but I can never remember any of the details of Feferman's treatment.

Can we somehow get rid of the conventional choices?

Coordinates: Solution 1A

One strategy of solution is mathematical abstraction: by analyzing Gödel's proof we find out what properties a predicate must have to make it work and declare anything satisfying these properties a proof predicate.

This leads to the Löb Conditions:

L1 $\vdash \phi \quad \Rightarrow \; \vdash \Box\phi$

L2 $\vdash (\Box\phi \wedge \Box(\phi \to \psi)) \to \Box\psi$

L3 $\vdash \Box\phi \to \Box\Box\phi$

The Löb conditions lead to provability logic. A subject Dick, Lev and I intensively studied.

Coordinates: Solution 1B

We still need a choice of numbers and a Gödel numbering to make sense of this. Currently, Balthasar Grabmayr is exploring general conditions for possible Gödel numberings.

False positives: There are predicates like $0 = 0$ that satisfy the conditions that are not much like provability.

False negatives: Conversely, even for our current formulation, some of the $\Box_\alpha$ that are perfectly legitimate proof predicates do not satisfy the Löb conditions.

Coordinates: Solution 2

We can try to characterized consistency statements by their 'powers'. Specifically, we have interpretability power: $(\mathsf{B} + \Diamond_\sigma \top) \rhd U$, where σ axiomatizes U.

Such attempts typically overgenerate.

We can improve the idea in various ways, pin-pointing consistency statements in a coordinate-free way modulo an equivalence relation.

To my taste, all such equivalence relations are still too coarse.

Alas, it would take too much time to sketch these ideas in more detail.

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Applications

G2 is more a basic insight than a general theorem with many applications. Yet it is still productive and delivering applications. Here are some examples.

- The Ryll Nardzewski Theorem: PA is not finitely axiomatizable. We even have: PA is not mutually interpretable with a finitely axiomatized theory.
- Comparison of Strength: ZF interprets PA but not vice versa.
- Speed up: Superexponential lower bounds for various speed-up results.
- Degree Structures: The degrees of interpretability have no co-atoms.
- Negation of Collection: Interpretability of the negation of collection over various theories.

Thank You

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