Notes on the computational aspects of Kripke's theory of truth

Stanislav O. Speranski

Associate Professor
St. Petersburg State University

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Kripke's Theory of Truth

Consider the signature of Peano arithmetic and its expansion obtained by adding an extra unary predicate symbol T, viz.

$$\sigma \;:=\; \{0, \mathrm{s}, +, \times, =\} \quad \text{and} \quad \sigma_T \;:=\; \sigma \cup \{T\}.$$

Throughout this presentation the following assumptions are in force:

- the connective symbols are $\neg$, $\wedge$ and $\vee$;
- the quantifier symbols are $\forall$ and $\exists$.

We abbreviate $\neg\varphi \vee \psi$ to $\varphi \to \psi$, $(\varphi \to \psi) \wedge (\psi \to \varphi)$ to $\varphi \leftrightarrow \psi$, etc. Let $\mathcal{L}$ and $\mathcal{L}_T$ be the first-order languages of σ and σ_T respectively.

Here is some related notation:

$$
\begin{aligned}
\mathit{For} &:= \text{the collection of all } \mathcal{L}\text{-formulas;} \\
\mathit{Sen} &:= \text{the collection of all } \mathcal{L}\text{-sentences;} \\
\mathit{For}_T &:= \text{the collection of all } \mathcal{L}_T\text{-formulas;} \\
\mathit{Sen}_T &:= \text{the collection of all } \mathcal{L}_T\text{-sentences.}
\end{aligned}
$$

Assume some Gödel numbering $\#$ of $\mathcal{L}_T$ has been chosen. Then we call $A \subseteq \mathbb{N}$ consistent iff there is no $\phi \in \mathit{Sen}_T$ s.t. both $\#\phi$ and $\#\neg\phi$ are in A. If $A \subseteq \mathbb{N}$, we write $\langle \mathfrak{N}, A \rangle$ for the expansion of the standard model $\mathfrak{N}$ of Peano arithmetic to σ_T in which T is interpreted as A.

In his 'Outline of a theory of truth', Kripke used partial interpretations of T, i.e. pairs of the form $S = \langle S^+, S^- \rangle$ where S^+ and S^- are disjoint subsets of $\mathbb{N}$, resp. called the extension of S and the anti-extension of S. Henceforth we limit ourselves to partial interpretations of T with consistent extensions. A partial valuation for σ_T is a mapping from Sen_T to a superset of $\left\{0, \frac{1}{2}, 1\right\}$.

By a valuation scheme we mean a function from partial interpretations to partial valuations. To begin with, let $\leqslant_{\mathrm{sK}}$ and $\leqslant_{\mathrm{wK}}$ be the orderings given by

$$0 \;\leqslant_{\mathrm{sK}}\; \tfrac{1}{2} \;\leqslant_{\mathrm{sK}}\; 1 \quad \text{and} \quad \tfrac{1}{2} \;\leqslant_{\mathrm{wK}}\; 0 \;\leqslant_{\mathrm{wK}}\; 1.$$

Define the strong Kleene valuation scheme V_{sK} inductively as follows:

- for any closed $\mathcal{L}$-terms t_1 and t_2,

$$V_{\mathrm{sK}}(S)(t_1 = t_2) := \begin{cases} 1 & \text{if } \mathfrak{N} \models t_1 = t_2, \\ 0 & \text{if } \mathfrak{N} \models t_1 \neq t_2; \end{cases}$$

- for every closed $\mathcal{L}$-term t,

$$V_{\mathrm{sK}}(S)(T(t)) := \begin{cases} 1 & \text{if } \langle \mathfrak{N}, S^+ \rangle \models T(t), \\ 0 & \text{if } \langle \mathfrak{N}, S^- \cup (\mathbb{N} \setminus \# Sen_T) \rangle \models T(t), \\ \frac{1}{2} & \text{otherwise}; \end{cases}$$

- $V_{\mathrm{sK}}(S)(\varphi \wedge \phi) := \min\limits_{\leqslant_{\mathrm{sK}}} \{ V_{\mathrm{sK}}(S)(\varphi), V_{\mathrm{sK}}(S)(\phi) \}$;
- $V_{\mathrm{sK}}(S)(\forall x\, \varphi(x)) := \min\limits_{\leqslant_{\mathrm{sK}}} \{ V_{\mathrm{sK}}(S)(\varphi(t)) \mid t \text{ is a closed } \mathcal{L}\text{-term} \}$;

- $V_{\mathrm{sK}}(S)(\varphi \vee \phi) := V_{\mathrm{sK}}(S)(\neg(\neg\varphi \wedge \neg\phi))$;
- $V_{\mathrm{sK}}(S)(\exists x\, \varphi(x)) := V_{\mathrm{sK}}(S)(\neg\forall x\, \neg\varphi(x))$;
- $V_{\mathrm{sK}}(S)(\neg\varphi) := 1 - V_{\mathrm{sK}}(S)(\varphi)$.

To get the weak Kleene valuation scheme V_{wK}, simply replace $\leqslant_{\mathrm{sK}}$ by $\leqslant_{\mathrm{wK}}$. Next we turn to so-called supervaluation schemes, each of which has the form

$$V(S)(\varphi) := \begin{cases} 1 & \text{if for all } A \subseteq \mathbb{N} \text{ satisfying } [*],\ \langle \mathfrak{N}, A\rangle \models \varphi, \\ 0 & \text{if for all } A \subseteq \mathbb{N} \text{ satisfying } [*],\ \langle \mathfrak{N}, A\rangle \models \neg\varphi, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

The best known such schemes are V_{SV}, V_{VB}, V_{FV} and V_{MC}, given by:

$$
\begin{aligned}
V &= V_{\mathrm{SV}} &\Longleftrightarrow\quad [*] &= \text{`}S^+ \subseteq A\text{'};\\
V &= V_{\mathrm{VB}} &\Longleftrightarrow\quad [*] &= \text{`}S^+ \subseteq A \text{ and } A \cap S^- = \varnothing\text{'};\\
V &= V_{\mathrm{FV}} &\Longleftrightarrow\quad [*] &= \text{`}S^+ \subseteq A \text{ and } A \text{ is consistent'};\\
V &= V_{\mathrm{MC}} &\Longleftrightarrow\quad [*] &= \text{`}S^+ \subseteq A \text{ and } A \text{ is cons. and complete'}.
\end{aligned}
$$

Here the 'completeness' of A means that for each $\phi \in Sen_T$ we have $\#\phi \in A$ or $\#\neg\phi \in A$. The last scheme emerges from Leitgeb's 'What truth depends on' (although the definition presented below was stated explicitly by his PhD student Thomas Schindler). Say that $\varphi \in Sen_T$ depends on $A \subseteq \mathbb{N}$ iff for every $B \subseteq \mathbb{N}$,

$$\langle \mathfrak{N}, B \rangle \models \varphi \quad \Longleftrightarrow \quad \langle \mathfrak{N}, B \cap A \rangle \models \varphi.$$

Now define Leitgeb's valuation scheme V_{L} by

$$V_{\mathrm{L}}(S)(\varphi) := \begin{cases} 1 & \text{if } \varphi \text{ depends on } S^+ \cup S^- \text{ and } \langle \mathfrak{N}, S^+ \rangle \models \varphi, \\ 0 & \text{if } \varphi \text{ depends on } S^+ \cup S^- \text{ and } \langle \mathfrak{N}, S^+ \rangle \models \neg\varphi, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

It should be noted that each valuation scheme V induces a function $\mathfrak{J}_V$ from partial interpretations to partial interpretations, called the Kripke-jump operator for V, as follows:

$$\begin{aligned} \mathfrak{J}_V(S)^+ &:= \{\#\varphi \mid \varphi \in Sen_T \text{ and } V(S)(\varphi) = 1\}, \\ \mathfrak{J}_V(S)^- &:= \{\#\varphi \mid \varphi \in Sen_T \text{ and } V(S)(\varphi) = 0\} \cup \\ &\qquad \cup \{n \in \mathbb{N} \mid n \notin \#Sen_T\}. \end{aligned}$$

In turn $\mathcal{J}_V$ generates a transfinite sequence indexed by ordinals:

$$\mathcal{J}_V^{\alpha}(S) := \begin{cases} S & \text{if } \alpha = 0, \\ \mathcal{J}_V\left(\mathcal{J}_V^{\beta}(S)\right) & \text{if } \alpha = \beta + 1, \\ \langle \bigcup_{\beta<\alpha} \mathcal{J}_V^{\beta}(S)^{+}, \bigcup_{\beta<\alpha} \mathcal{J}_V^{\beta}(S)^{-} \rangle & \text{if } \alpha \in \text{L-Ord}. \end{cases}$$

We shall often write T_V^{α} instead of $\mathcal{J}_V^{\alpha}(\varnothing, \varnothing)^{+}$ — these sets constitute the truth hierarchy for V.

Moreover Kripke dealt with monotone schemes, i.e. those which satisfy the condition that for any partial interpretations S_1 and S_2,

$$S_1^{+} \subseteq S_2^{+} \ \& \ S_1^{-} \subseteq S_2^{-} \implies$$
$$\implies \mathcal{J}_V(S_1)^{+} \subseteq \mathcal{J}_V(S_2)^{+} \ \& \ \mathcal{J}_V(S)^{-} \subseteq \mathcal{J}_V(S)^{-}.$$

Observation (Kripke)

For every monotone valuation scheme V there exists an ordinal α s.t. $\mathrm{T}_V^{\alpha} = \mathrm{T}_V^{\alpha+1}$ — yielding the least fixed point of $\mathfrak{J}_V$.

It is easy to verify that each $V \in \{V_{\mathrm{sK}}, V_{\mathrm{wK}}, V_{\mathrm{SV}}, V_{\mathrm{VB}}, V_{\mathrm{FV}}, V_{\mathrm{MC}}, V_{\mathrm{L}}\}$ is monotone and furthermore has the following properties:

- if $\mathfrak{J}_V(S) = S$, then $V(S)(T(\ulcorner\varphi\urcorner)) = V(S)(\varphi)$;
- $\#\varphi \in \mathfrak{J}_V^{\alpha}(S)^{-}$ iff $\#\neg\varphi \in \mathfrak{J}_V^{\alpha}(S)^{+}$;
- $\#\varphi \in \mathfrak{J}_V^{\alpha}(S)^{+}$ iff $\#\neg\varphi \in \mathfrak{J}_V^{\alpha}(S)^{-}$;
- $\mathfrak{J}_V$ turns out to be a 'Π_1^1-operator' — so by a well-known theorem of Spector, $\mathrm{T}_V^{\alpha} = \mathrm{T}_V^{\alpha+1}$ already for some $\alpha \in$ C-Ord $\cup \{\omega_1^{\mathrm{CK}}\}$.

Kleene's $\mathcal{O}$

Remember that Kleene's system of notation for C-Ord consists of:

- a special partial function $\nu_{\mathcal{O}}$ from $\mathbb{N}$ onto C-Ord;
- an appropriate ordering relation $<_{\mathcal{O}}$ on $\mathrm{dom}\,(\nu_{\mathcal{O}})$ — which mimics the usual ordering relation on C-Ord.

Call $n \in \mathbb{N}$ a notation for $\alpha \in$ C-Ord iff $\nu_{\mathcal{O}}\,(n) = \alpha$. To simplify the statements I often write $n \in \mathcal{O}$ instead of $n \in \mathrm{dom}\,(\nu_{\mathcal{O}})$.

Folklore

$\mathrm{dom}\,(\nu_{\mathcal{O}})$ *is* Π^1_1*-complete.*

Fix one's favorite universal partial computable (two-place) function U.

Folklore

There exists a computable function f such that for every $n \in \mathcal{O}$,

$$\{k \in \mathbb{N} \mid k <_{\mathcal{O}} n\} \;=\; \operatorname{dom}\left(U_{f(n)}\right).$$

Folklore (Effective Transfinite Recursion)

Suppose f is a computable function such that for any $e \in \mathbb{N}$ and $n \in \mathcal{O}$,

$$\{k \in \mathbb{N} \mid k <_{\mathcal{O}} n\} \;\subseteq\; \operatorname{dom}\left(U_e\right) \quad\Longrightarrow\quad n \;\in\; \operatorname{dom}\left(U_{f(e)}\right).$$

Then there is a $c \in \mathbb{N}$ for which $U_{f(c)} = U_c$, and $\operatorname{dom}\left(\nu_{\mathcal{O}}\right) \subseteq \operatorname{dom}\left(U_c\right)$.

About Least Fixed-Points

Let us call a valuation scheme V ordinary iff for any $\alpha \in \mathrm{Ord}$, $\chi \in Sen$, $\psi \in Sen_T$ and $\varphi(x) \in For_T$ the following conditions hold:

1. $\mathrm{T}_V^{\alpha} \subseteq \mathrm{T}_V^{\alpha+1}$;
2. $\chi \in \mathrm{T}_V^{\alpha}$ iff $\alpha \neq 0$ and $\mathfrak{N} \models \chi$;
3. $\psi \in \mathrm{T}_V^{\alpha}$ iff $T(\ulcorner\psi\urcorner) \in \mathrm{T}_V^{\alpha+1}$;
4. $\forall x\, \varphi(x) \in \mathrm{T}_V^{\alpha+1}$ iff $\{\varphi(\underline{n}) \mid n \in \mathbb{N}\} \subseteq \mathrm{T}_V^{\alpha+1}$;
5. $\chi \wedge \psi \in \mathrm{T}_V^{\alpha}$ iff $\mathfrak{N} \models \chi$ and $\psi \in \mathrm{T}_V^{\alpha}$;
6. if $\chi \vee \psi \in \mathrm{T}_V^{\alpha}$ and $\mathfrak{N} \models \neg\chi$, then $\psi \in \mathrm{T}_V^{\alpha}$;
7. if $\mathfrak{N} \models \chi$ and $\alpha \neq 0$, then $\chi \vee \psi \in \mathrm{T}_V^{\alpha}$.

In effect, except for V_{wK}, all the schemes considered above are ordinary.

Given V, by the rank of $\psi \in Sen_T$ — denoted by $rank_V(\psi)$ — I mean the least ordinal α for which $\psi \in \mathrm{T}_V^{\alpha+1}$.

Proposition

Let V be a valuation scheme satisfying (3–4). Then for every $\psi \in Sen_T$ and every $\varphi(x) \in For_T$,

$$\begin{aligned} rank_V(T(\ulcorner \psi \urcorner)) &= rank_V(\psi) + 1 \quad \textit{and} \\ rank_V(\forall x\, \varphi(x)) &= \sup\{rank_V(\varphi(\underline{n})) \mid n \in \mathbb{N}\}. \end{aligned}$$

Proposition ♮

For each ordinary scheme V there exists a computable function ρ_V such that for every $n \in \mathcal{O}$, $rank_V(\rho_V(n)) = \nu_{\mathcal{O}}(n) + 1$.

Corollary ♯

Each ordinary scheme V has the following property: for every ordinal α, if $\mathrm{T}_V^{\alpha} = \mathrm{T}_V^{\alpha+1}$, then $\alpha \geqslant \omega_1^{\mathrm{CK}}$ and T_V^{α} is Π_1^1-hard.

The technique used in the proofs of these facts can be applied in various other situations as well. Let us see how it works e.g. for V_{wK}. Still, as it was shown by Cain and Damnjanovic, one should be warned:

> *Actually certain complexity results for the weak Kleene scheme depend on the Gödel numbering and the language of the "standard" model of arithmetic we choose.*

I am aiming at a deeper understanding of this intensionality phenomenon.

In their article from 1991, Cain and Damnjanovic suggested expanding σ to avoid the conflict. More precisely, assuming an appropriate coding $\mathrm{M}_0, \mathrm{M}_1, \ldots$ of all Turing machines, they added a new function symbol π of arity 4, whose interpretation is given by

$$\pi(e, i, j, k) \; := \; \begin{cases} n & \text{if } \mathrm{M}_e \text{ halts on input } i \text{ at step } j \text{ with output } n, \\ k & \text{if } \mathrm{M}_e \text{ does not halt on input } i \text{ at step } j. \end{cases}$$

Clearly this function is primitive recursive. *So what can we do with* π*?*

Observation $\diamondsuit$

If we include π *in* σ*, then both Proposition* $\natural$ *and Corollary* $\sharp$ *generalise to arbitrary valuation schemes satisfying (1–5).*

As an alternative to Cain–Damnjanovic' suggestion, I propose to add a symbol $\dot{-}$ for cut-off subtraction, i.e. $i \dot{-} j := \max\{0, i - j\}$:

Observation ♡

Similar to Observation ♢, but with $\dot{-}$ instead of π.

Another modification, with $\mathcal{L}$ unchanged, deals with the following condition (for any $\alpha \in \mathrm{Ord}$, $\theta(x) \in For$ and $\varphi(x) \in For_T$):

8. $\exists x\,(\theta(x) \wedge T(x)) \in \mathrm{T}^{\alpha}_{V}$ iff $\mathfrak{N} \models \theta(\underline{n})$ and $T(\underline{n}) \in \mathrm{T}^{\alpha}_{V}$ for some $n \in \mathbb{N}$.

Observation ♠

The analogues of Proposition ♮ and Corollary ♯ hold for all valuation schemes satisfying (1–5) and (8).

In fact, although (8) fails for the weak Kleene scheme, the customary treatment of $\exists$ in the case of V_{wK} does not seem to be well motivated. Alternatively, we can define V^{*}_{wK} exactly as V_{wK} except that

$$V^{*}_{\mathrm{wK}}(S)(\exists x\,\varphi(x)) \;:=\; \max_{\leqslant_{\mathrm{sK}}}\{V^{*}_{\mathrm{wK}}(S)(\varphi(t)) \mid t \text{ is a closed } \mathcal{L}\text{-term}\}.$$

(like in the strong Kleene scheme V_{sK}). Then V^{*}_{wK} satisfies (1–5) and (8), so Observation ♠ applies. Furthermore, one could think of $\vee$ as a special case of $\exists$, which leads to

$$V^{*}_{\mathrm{wK}}(S)(\varphi \vee \phi) \;:=\; \max_{\leqslant_{\mathrm{sK}}}\{V^{*}_{\mathrm{wK}}(S)(\varphi), V^{*}_{\mathrm{wK}}(S)(\phi)\}.$$

This would give an ordinary scheme, so (8) would not be even needed.

Earlier we took $\to$ as an abbreviation. However, interpreting $\varphi \to \psi$ as $\neg\varphi \vee \psi$ is not always the right choice. To avoid confusion, I add a new connective symbol $\twoheadrightarrow$ to the original three (viz. $\neg$, $\wedge$ and $\vee$). Of course *For*, *For*$_T$, *Sen* and *Sen*$_T$ are easily modified to accommodate $\twoheadrightarrow$. Now consider the following variation on (6–7):

6′ if $\chi \twoheadrightarrow \psi \in \mathrm{T}^{\alpha}_{V}$ and $\mathfrak{N} \models \chi$, then $\psi \in \mathrm{T}^{\alpha}_{V}$;

7′ if $\mathfrak{N} \models \neg\chi$, then $\chi \twoheadrightarrow \psi \in \mathrm{T}^{\alpha}_{V}$

— where χ and ψ range over the modified versions of *Sen* and *Sen*$_T$ resp. Evidently, even when we treat $\twoheadrightarrow$ as the material conditional on $\{0, 1\}$, the meanings of $\varphi \twoheadrightarrow \psi$ and $\neg\varphi \vee \psi$ may differ on $\{0, \frac{1}{2}, 1\}$.

Observation ♣

If we expand $\mathcal{L}$ and $\mathcal{L}_T$ by adding $\twoheadrightarrow$, then the analogues of Proposition ♮ and Corollary ♯ hold for all valuation schemes satisfying (1–5) and (6'–7').

This is closely related to a three-valued scheme from Feferman's article 'Axioms for determinateness and truth'. It can be obtained by extending V_{wK} to formulas containing $\twoheadrightarrow$ by setting

$$V'_{\mathrm{wK}}(S)(\varphi \twoheadrightarrow \psi) \;:=\; \max_{\leqslant_{\mathrm{sK}}} \{V_{\mathrm{wK}'}(S)(\neg\varphi), V_{\mathrm{wK}'}(S)(\varphi \wedge \psi)\};$$

(the other clauses are the same as in the definition of V_{wK}). Now V'_{wK} satisfies (1–5) and (6'–7'), so Observation ♣ applies.

Note that every $\mathcal{L}_T$-formula can be viewed as an arithmetical monadic second-order $\sigma_{\mathbb{N}}$-formula whose only set variable is T, and vice versa. Given an $\mathcal{L}_T$-sentence ψ and an $\mathcal{L}$-formula $\chi(x)$, we construct

$$\psi_\chi \; := \; \text{the result of replacing each } T(t) \text{ in } \psi \text{ by } \chi(t) \wedge T(t).$$

Observation ✠

Let $\chi(x)$ be an $\mathcal{L}$-formula defining an infinite computable subset of $\mathbb{N}$ in $\mathfrak{N}$. Then $\{\psi_\chi \mid \psi \in Sen_T \text{ and } \mathfrak{N} \models \forall T\, \psi_\chi(T)\}$ is Π^1_1-complete.

It gives an alternative and probably the shortest proof for the following.

Theorem (Welch, Hjorth, Meadows)

For any $V \in \{V_{\mathrm{SV}}, V_{\mathrm{VB}}, V_{\mathrm{FV}}, V_{\mathrm{MC}}, V_{\mathrm{L}}\}$ and $\alpha \in \mathrm{Ord}^+$, T^α_V is Π^1_1-hard.

Proof.

Assume $V = V_{\mathrm{L}}$. Take A to be $\#\{\mu, T(\ulcorner\mu\urcorner), T(\ulcorner T(\ulcorner\mu\urcorner)\urcorner), \dots\}$ with μ denoting some fixed 'truthteller', and let χ be an $\mathcal{L}$-formula defining A in $\mathfrak{N}$. Since $A \cap \mathrm{G} = \varnothing$, we obtain

$$\begin{aligned} \#\psi_\chi \in \mathrm{T}_V^{\beta+1} &\iff \#\psi_\chi \in \mathrm{G}_{\beta+1} \text{ and } \langle \mathfrak{N}, \mathrm{T}_V^\beta \rangle \models \psi_\chi \\ &\iff \mathfrak{N} \models \forall T\,(\psi_\chi\,(T \cap \mathrm{G}_\beta) \leftrightarrow \psi_\chi\,(T)) \land \psi_\chi(\mathrm{T}_V^\beta) \\ &\iff \mathfrak{N} \models \forall T\,(\psi_\chi\,(\varnothing) \leftrightarrow \psi_\chi\,(T)) \land \psi_\chi(\varnothing) \\ &\iff \mathfrak{N} \models \forall T\,\psi_\chi\,(T). \end{aligned}$$

Clearly $\mathrm{T}_V^\alpha = \bigcup_{\beta<\alpha} \mathrm{T}_V^{\beta+1}$, so the Π_1^1-hardness of T_V^α follows by Observation ✠. Perfectly analogous arguments apply to the other schemes. □

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