Non-Well-Founded Proofs for Modal Grzegorczyk Logic

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Grzegorczyk logic

The Grzegorczyk logic Grz is a well-known modal logic, which can be characterized by reflexive partially ordered Kripke frames without infinite ascending chains.

This logic is complete w.r.t. the arithmetical semantics, where the modal connective $\Box$ corresponds to the strong provability operator "*... is true and provable*" in Peano arithmetic.

The Hilbert-style axiomatization of Grz

Axioms:

(i) Boolean tautologies;

(ii) $\Box(A \to B) \to (\Box A \to \Box B)$;

(iii) $\Box A \to \Box\Box A$;

(iv) $\Box A \to A$;

(v) $\Box(\Box(A \to \Box A) \to A) \to \Box A$ (Grzegorczyk axiom).

Rules: modus ponens, $A/\Box A$.

The system $\mathsf{Grz}_{\mathsf{Seq}}$

$$\Gamma, A \Rightarrow A, \Delta \;, \qquad\qquad \Gamma, \bot \Rightarrow \Delta \;,$$

$$\to_{\mathsf{L}} \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta} \;, \quad \to_{\mathsf{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta} \;,$$

$$\mathsf{refl} \frac{\Gamma, B, \Box B \Rightarrow \Delta}{\Gamma, \Box B \Rightarrow \Delta} \;, \qquad \Box_{\mathsf{Grz}} \frac{\Box\Pi, \Box(A \to \Box A) \Rightarrow A}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta} \;.$$

The cut rule

The cut rule has the form

$$\text{cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta},$$

where A is called the *cut formula* of the given inference.

Lemma

$\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathsf{Grz} \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$.

Proof.

Standard transformations of proofs. □

Cut Elimination

Theorem

If $\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Gamma \Rightarrow \Delta$.

Cut elimination for the (variant) of the system $\mathsf{Grz}_{\mathsf{Seq}}$ was proven syntactically by Borga and Gentilini in 1986.

How do We Prove Cut Elimination?

1. Translate into a system with non-well-founded proofs.
   Why do we want to do that?
2. Eliminate cut in that system.
   How do we do that on infinite proofs?
3. Translate it back.
   How do we obtain a finite proof from an infinite (not even cyclical!) one?

The system Grz_∞

$$\Gamma, p \Rightarrow p, \Delta \ , \qquad \Gamma, \bot \Rightarrow \Delta \ ,$$

$$\to_{\mathsf{L}} \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta} \ , \qquad \to_{\mathsf{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta} \ ,$$

$$\mathsf{refl} \frac{\Gamma, A, \Box A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \ , \qquad \Box \frac{\Gamma, \Box\Pi \Rightarrow A, \Delta \qquad \Box\Pi \Rightarrow A}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta} \ .$$

The system $\mathsf{Grz}_\infty + \mathsf{cut}$ is defined by adding the rule (cut) to the system Grz_∞.

Non-Well-Founded Proofs

An ∞–*proof* in Grz_∞ ($\mathsf{Grz}_\infty + \mathsf{cut}$) is a (possibly infinite) tree whose nodes are marked by sequents and whose leaves are marked by initial sequents and that is constructed according to the rules of the sequent calculus.

In addition, every infinite branch in an ∞–proof must pass through a right premise of the rule $\Box$ infinitely many times.

Main Fragment

The *main fragment* of an ∞–proof is a finite tree obtained from the ∞–proof by cutting every infinite branch at the nearest to the root right premise of the rule $(\Box)$.

The *local height* $|\pi|$ *of an* ∞*–proof* π is the length of the longest branch in its main fragment. An ∞–proof only consisting of an initial sequent has height 0.

Example

$$
\text{refl}\;\dfrac{
\to_{\mathrm{L}}\;\dfrac{
\dfrac{\text{Ax}}{F, p \Rightarrow p}
\qquad
\Box\;\dfrac{
\to_{\mathrm{R}}\;\dfrac{\dfrac{\text{Ax}}{F, p \Rightarrow \Box p, p}}{F \Rightarrow p \to \Box p, p}
\qquad
\to_{R}\;\dfrac{
\Box\;\dfrac{\dfrac{\text{Ax}}{p, F \Rightarrow p} \qquad \dfrac{\vdots}{F \Rightarrow p}}{p, F \Rightarrow \Box p}
}{F \Rightarrow p \to \Box p}
}{F \Rightarrow \Box(p \to \Box p), p}
}{\Box(p \to \Box p) \to p, F \Rightarrow p}
}{F \Rightarrow p},
$$

where $F = \Box(\Box(p \to \Box p) \to p)$.

The local height of this ∞–proof equals to 4.

Translation

The formula on the previous slide is basically the Grzegorczyk axiom, so we have

Theorem

If $\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{Grz}_{\infty} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$.

Cut Elimination: How Gentzen Did It

Gentzen cut-elimination procedure can be viewed as operator that "commutes" with all the rules but cut, and satisfies the following equation:

$$\mathrm{u}(\mathrm{cut}\frac{\pi_0 \qquad \pi_1}{\Gamma \Rightarrow \Delta}) = \mathrm{re}_A(\mathrm{u}(\pi_0), \mathrm{u}(\pi_1)).$$

where re_A is the operation that takes cut-free proofs of $\Gamma \Rightarrow A, \Delta$ and $\Gamma, A \Rightarrow \Delta$ and returns a cut-free proof of $\Gamma \Rightarrow \Delta$.

For the systems with finite proofs and subformula property re_A can be defined inductively.

How Do We Do It For Infinite Proofs?

It turns out, that our ∞-proofs forms a spherically complete ultrametric space. And so do root-preserving operators on proofs if you choose a right metric.

This allows us to define re_A and our cut-eliminating operator ce as fix-points of some non-expansive mappings on the set of root-preserving operators on proofs.

What Do We Need to Succeed

First we define the operators re_A in such a way that it does not "introduce" new cuts.

Once we have this we can define an operator F on the set of root-preserving operators on proofs the following way: for any ∞-proof π and a root-preserving operator u we define $\mathsf{F}(\mathsf{u})(\pi)$ is the result of applying the last rule (if it is not cut) in π to the results of applying u to the premises of it and put

$$\mathsf{F}(\mathsf{u})(\mathsf{cut}\frac{\pi_0 \quad\quad \pi_1}{\Gamma \Rightarrow \Delta}) = \mathsf{re}_A(\mathsf{u}(\pi_0), \mathsf{u}(\pi_1)).$$

Taking the Fix-Point

With our metric, the operator F turns out to be non-expansive, so it has a fix-point that we denote ce.

We then prove that for any ∞-proof π, the ∞-proof ce(π) indeed does not have any cuts.

Translating Back

For a sequent $\Gamma \Rightarrow \Delta$, let $Sub(\Gamma \Rightarrow \Delta)$ be the set of all subformulas of the formulas from $\Gamma \cup \Delta$.

For a finite set of formulas Λ, set $\Lambda^* := \{\Box(A \to \Box A) \mid A \in \Lambda\}$.

Lemma

If $\mathsf{Grz}_\infty \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Lambda^*, \Gamma \Rightarrow \Delta$ *for any finite set of formulas* Λ*.*

Proof

By induction on the number of elements in the finite set $Sub(\Gamma \Rightarrow \Delta) \backslash \Lambda$ with a subinduction on the local height of the proof.

The only non-trivial case is when our proof π has the form

$$\Box \frac{\begin{array}{c}\pi' \\ \Phi, \Box\Pi \Rightarrow A, \Sigma\end{array} \qquad \begin{array}{c}\pi'' \\ \Box\Pi \Rightarrow A\end{array}}{\Phi, \Box\Pi \Rightarrow \Box A, \Sigma} \ ,$$

where $\Phi, \Box\Pi = \Gamma$ and $\Box A, \Sigma = \Delta$.

Subcase 1

The formula A belongs to Λ.
We see that $|\pi'| < |\pi|$. By the induction hypothesis for π' and Λ, the sequent $\Lambda^*, \Phi, \Box\Pi \Rightarrow A, \Sigma$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Then we see

$$\mathrm{refl}\,\dfrac{\to_{\mathrm{L}}\,\dfrac{\dfrac{\mathrm{Ax}}{\Lambda^*, \Box A, \Phi, \Box\Pi \Rightarrow \Box A, \Sigma} \qquad \mathrm{weak}\,\dfrac{\Lambda^*, \Phi, \Box\Pi \Rightarrow A, \Sigma}{\Lambda^*, \Phi, \Box\Pi \Rightarrow A, \Box A, \Sigma}}{(\Lambda\backslash\{A\})^*, A \to \Box A, \Box(A \to \Box A), \Phi, \Box\Pi \Rightarrow \Box A, \Sigma}}{(\Lambda\backslash\{A\})^*, \Box(A \to \Box A), \Phi, \Box\Pi \Rightarrow \Box A, \Sigma},$$

where the rule (weak) is admissible.

Subcase 2

The formula A doesn't belong to Λ.
We have that the number of elements in $Sub(\Box\Pi \Rightarrow A) \backslash (\Lambda \cup \{A\})$ is strictly less than the number of elements in $Sub(\Phi, \Box\Pi \Rightarrow \Box A, \Sigma) \backslash \Lambda$. Therefore, by the induction hypothesis for π'' and $\Lambda \cup \{A\}$, the sequent $\Lambda^*, \Box(A \to \Box A), \Box\Pi \Rightarrow A$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Then we have

$$\Box_{\mathsf{Grz}} \frac{\Lambda^*, \Box(A \to \Box A), \Box\Pi \Rightarrow A}{\Lambda^*, \Phi, \Box\Pi \Rightarrow \Box A, \Sigma} .$$

$\Box$

Conclusion

- We can define logics by proof structure.
- For some purposes the subformula property is more helpful than finitness of proofs.
- Topology can be very helpful — the only thing we need is a right metric.

Future Work

- Syntactically prove different properties like Lyndon interpolation.
- Cyclical proofs.
- Different logics (Go, $\Box^{+}$).

Fin

Thank You!