Negative Church's Thesis and Russian Constructivism

SATO Kentaro

sato@inf.unibe.ch

Institution for Computer Science, University of Bern

* partially supported by John Templeton Foundation

Plan of This Talk

- Russian Recursive Constructive Mathematics (vs. Classical Mathematics and vs. Brouwer's Intuitionism);
- New Principle: Negative Church's Thesis NCT — representing RRCM-spirit better than CT?
- Realizability model of $\mathsf{NCT} + \mathsf{BCP} + \mathsf{MP}$ and "any $f : \mathbb{R} \to \mathbb{R}$ is continuous";
- Consequences of $\mathsf{NCT} + \mathsf{MP}$.

Plan of This Talk

- Russian Recursive Constructive Mathematics (vs. Classical Mathematics and vs. Brouwer's Intuitionism);
- New Principle: Negative Church's Thesis NCT — representing RRCM-spirit better than CT?
- Realizability model of $\mathsf{NCT} + \mathsf{BCP} + \mathsf{MP}$ and "any $f : \mathbb{R} \to \mathbb{R}$ is continuous";
- Consequences of $\mathsf{NCT} + \mathsf{MP}$.

All the technical results are proved in:

T. Nemoto and K. Sato
"A marriage of Brouwer's Intuitionism and Hilbert's Finitism I: Arithmetic",
to appear in *The Journal of Symbolic Logic*

Varieties of Mathematics

Classical Mathematics

Brouwer's Intuitionistic Mathematics

Varieties of Mathematics

Classical Mathematics

Brouwer's Intuitionistic Mathematics

- the precursor of (serious) constructivism;

Varieties of Mathematics

Classical Mathematics

Brouwer's Intuitionistic Mathematics

- the precursor of (serious) constructivism;
- rejection of LEM (intuitionistic logic);

Varieties of Mathematics

Classical Mathematics

Brouwer's Intuitionistic Mathematics

- the precursor of (serious) constructivism;
- rejection of LEM (intuitionistic logic);
- infinite sequences can be captured only via finite fragments
  $\rightarrow$ <u>Brouwer's continuous principle</u>;
- Bar Induction.

Varieties of Mathematics

Classical Mathematics

Brouwer's Intuitionistic Mathematics

- the precursor of (serious) constructivism;
- rejection of LEM (intuitionistic logic);
- infinite sequences can be captured only via finite fragments
  $\rightarrow$ <u>Brouwer's continuous principle</u>;
- Bar Induction.

Russian Recursive Constructive Mathematics

- rejection of LEM (intuitionistic logic);

Varieties of Mathematics

Classical Mathematics

Brouwer's Intuitionistic Mathematics

- the precursor of (serious) constructivism;
- rejection of LEM (intuitionistic logic);
- infinite sequences can be captured only via finite fragments
  $\rightarrow$ Brouwer's continuous principle;
- Bar Induction.

Russian Recursive Constructive Mathematics

- rejection of LEM (intuitionistic logic);
- everything is computable/recursive;

Varieties of Mathematics

Classical Mathematics

Brouwer's Intuitionistic Mathematics

- the precursor of (serious) constructivism;
- rejection of LEM (intuitionistic logic);
- infinite sequences can be captured only via finite fragments
  $\rightarrow$ Brouwer's continuous principle;
- Bar Induction.

Russian Recursive Constructive Mathematics

- rejection of LEM (intuitionistic logic);
- everything is computable/recursive;
- Markov's Principle: $\neg\neg\exists x \varphi[x] \rightarrow \exists x \varphi[x]$, for decidable φ.

Standard Formalization of RRCM

0. Heyting Arithmetic **HA** plus

Standard Formalization of RRCM

0. Heyting Arithmetic **HA** plus

1. Markov's Principle MP:
   - $\neg\neg\exists x \varphi[x] \to \exists x \varphi[x]$ for a Δ_0 formula $\varphi[x]$ or
   - $\forall x(\varphi(x) \vee \neg\varphi[x]) \to (\neg\neg\exists x \varphi[x] \to \exists x \varphi[x])$ for *any* formula $\varphi[x]$;

Standard Formalization of RRCM

0. Heyting Arithmetic **HA** plus

1. Markov's Principle MP:
   - $\neg\neg\exists x \varphi[x] \to \exists x \varphi[x]$ for a Δ_0 formula $\varphi[x]$
     or
   - $\forall x(\varphi(x) \vee \neg\varphi[x]) \to (\neg\neg\exists x \varphi[x] \to \exists x \varphi[x])$
     for *any* formula $\varphi[x]$;

2. and Church's Thesis:
   - $\forall x \exists y \varphi[x, y] \to \exists e \forall x(\{e\}(x)\downarrow \wedge \varphi[x, \{e\}(x)])$
     or
   - $\forall \alpha \exists e \forall x(\{e\}(x)\downarrow \wedge \{e\}(x) = \alpha(x))$
     $+$ choice $\forall x \exists y \varphi[x, y] \to \exists \alpha \forall x \varphi[x, \alpha(x)]$
     (in the function-based 2nd order setting).

Kleene's Number Realizability

For a formula φ, define another formula $e \,\mathfrak{nr}\, \varphi$ by

$$
\begin{aligned}
e \,\mathfrak{nr}\, \varphi &\equiv \varphi \quad \text{for atomic } \varphi; \\
e \,\mathfrak{nr}\, \varphi \wedge \psi &\equiv ((e)_0 \,\mathfrak{nr}\, \varphi) \wedge ((e)_1 \,\mathfrak{nr}\, \psi); \\
e \,\mathfrak{nr}\, \varphi \vee \psi &\equiv ((e)_0 = 0 \wedge (e)_1 \,\mathfrak{nr}\, \varphi) \vee ((e)_0 \neq 0 \wedge (e)_1 \,\mathfrak{nr}\, \psi); \\
e \,\mathfrak{nr}\, \varphi \to \psi &\equiv \forall x((x \,\mathfrak{nr}\, \varphi) \to (\{e\}(x)\!\downarrow \wedge \{e\}(x) \,\mathfrak{nr}\, \psi)); \\
e \,\mathfrak{nr}\, \exists x \varphi[x] &\equiv (e)_1 \,\mathfrak{nr}\, \varphi[(e)_0]; \\
e \,\mathfrak{nr}\, \forall x \varphi[x] &\equiv \forall x(\{e\}(x)\!\downarrow \wedge \{e\}(x) \,\mathfrak{nr}\, \varphi[x]).
\end{aligned}
$$

Kleene's Number Realizability

For a formula φ, define another formula $e\,\mathfrak{nr}\,\varphi$ by

$$e\,\mathfrak{nr}\,\varphi \equiv \varphi \quad \text{for atomic } \varphi;$$

$$e\,\mathfrak{nr}\,\varphi \wedge \psi \equiv ((e)_0\,\mathfrak{nr}\,\varphi) \wedge ((e)_1\,\mathfrak{nr}\,\psi);$$

$$e\,\mathfrak{nr}\,\varphi \vee \psi \equiv ((e)_0 = 0 \wedge (e)_1\,\mathfrak{nr}\,\varphi) \vee ((e)_0 \neq 0 \wedge (e)_1\,\mathfrak{nr}\,\psi);$$

$$e\,\mathfrak{nr}\,\varphi \rightarrow \psi \equiv \forall x((x\,\mathfrak{nr}\,\varphi) \rightarrow (\{e\}(x)\!\downarrow \wedge \{e\}(x)\,\mathfrak{nr}\,\psi));$$

$$e\,\mathfrak{nr}\,\exists x\varphi[x] \equiv (e)_1\,\mathfrak{nr}\,\varphi[(e)_0];$$

$$e\,\mathfrak{nr}\,\forall x\varphi[x] \equiv \forall x(\{e\}(x)\!\downarrow \wedge \{e\}(x)\,\mathfrak{nr}\,\varphi[x]).$$

A straightforward formalization of BHK interpretation by "algorithms" = "partial recursive functions".

Kleene's Number Realizability

For a formula φ, define another formula $e\,\mathfrak{nr}\,\varphi$ by

$$e\,\mathfrak{nr}\,\varphi \equiv \varphi \quad \text{for atomic } \varphi;$$

$$e\,\mathfrak{nr}\,\varphi \wedge \psi \equiv ((e)_0\,\mathfrak{nr}\,\varphi) \wedge ((e)_1\,\mathfrak{nr}\,\psi);$$

$$e\,\mathfrak{nr}\,\varphi \vee \psi \equiv ((e)_0 = 0 \wedge (e)_1\,\mathfrak{nr}\,\varphi) \vee ((e)_0 \neq 0 \wedge (e)_1\,\mathfrak{nr}\,\psi);$$

$$e\,\mathfrak{nr}\,\varphi \to \psi \equiv \forall x((x\,\mathfrak{nr}\,\varphi) \to (\{e\}(x)\!\downarrow \wedge \{e\}(x)\,\mathfrak{nr}\,\psi));$$

$$e\,\mathfrak{nr}\,\exists x \varphi[x] \equiv (e)_1\,\mathfrak{nr}\,\varphi[(e)_0];$$

$$e\,\mathfrak{nr}\,\forall x \varphi[x] \equiv \forall x(\{e\}(x)\!\downarrow \wedge \{e\}(x)\,\mathfrak{nr}\,\varphi[x]).$$

A straightforward formalization of BHK interpretation by "algorithms" = "partial recursive functions".

But "BHK interpretation"="Brouwer's world"?

Standard Formalization of BIM

Brouwer's Intuitionistic Mathematics is formalized, within the function-based 2nd order language, by

0. Heyting Arithmetic **HA** plus
1. Induction: for any formula φ
   $\varphi[0] \wedge \forall x(\varphi[x] \to \varphi[x+1]) \to \forall x \varphi[x]$;

Standard Formalization of BIM

Brouwer's Intuitionistic Mathematics is formalized, within the function-based 2nd order language, by

0. Heyting Arithmetic **HA** plus
1. Induction: for any formula φ
   $\varphi[0] \wedge \forall x(\varphi[x] \to \varphi[x+1]) \to \forall x \varphi[x]$;
2. Axiom of Choice AC: for any formula φ
   $\forall x \exists y \varphi[x, y] \to \exists \alpha \forall x \varphi[x, \alpha(x)]$;

Standard Formalization of BIM

Brouwer's Intuitionistic Mathematics is formalized, within the function-based 2nd order language, by

0. Heyting Arithmetic **HA** plus
1. Induction: for any formula φ
   $\varphi[0] \wedge \forall x(\varphi[x] \to \varphi[x{+}1) \to \forall x \varphi[x]$;
2. Axiom of Choice AC: for any formula φ
   $\forall x \exists y \varphi[x, y] \to \exists \alpha \forall x \varphi[x, \alpha(x)]$;
3. Brouwer Continuous Principle: for any formula φ
   $\forall \alpha \exists x \varphi[\alpha, x] \to \forall \alpha \exists n, x \forall \beta(\beta \restriction n = \alpha \restriction n \to \varphi[\beta, x])$;

Standard Formalization of BIM

Brouwer's Intuitionistic Mathematics is formalized, within the function-based 2nd order language, by

0. Heyting Arithmetic **HA** plus
1. Induction: for any formula φ
   $\varphi[0] \wedge \forall x(\varphi[x] \to \varphi[x+1]) \to \forall x \varphi[x]$;
2. Axiom of Choice AC: for any formula φ
   $\forall x \exists y \varphi[x, y] \to \exists \alpha \forall x \varphi[x, \alpha(x)]$;
3. Brouwer Continuous Principle: for any formula φ
   $\forall \alpha \exists x \varphi[\alpha, x] \to \forall \alpha \exists n, x \forall \beta(\beta \restriction n = \alpha \restriction n \to \varphi[\beta, x])$;
4. Brouwer's Bar Induction: for a Δ^0_0 formula φ
   $\forall \alpha \exists n \varphi[\alpha \restriction n] \wedge \forall u(\forall x \varphi[u * \langle x \rangle] \to \varphi[u]) \to \varphi[\langle\, \rangle]$

Church's Thesis Contradicts BCP

- Church's Thesis:
  $\forall\alpha\exists e(\alpha = \{e\})$.
- Brouwer's Continuity Principle implies:
  $\forall\alpha\exists n\forall\beta(\beta{\upharpoonright}n = \alpha{\upharpoonright}n \rightarrow \beta = \{e\})$.

Church's Thesis Contradicts BCP

- Church's Thesis:
  $\forall\alpha\exists e(\alpha = \{e\})$.
- Brouwer's Continuity Principle implies:
  $\forall\alpha\exists n\forall\beta(\beta\restriction n = \alpha\restriction n \to \beta = \{e\})$.
- Particularly, $\exists n\forall\beta(\beta\restriction n = \underline{0}\restriction n \to \beta = \{e\})$.

Different Treatments of Reals

The contradiction seems to be on the difference:

- In Brouwer's idea:

  infinite sequences can be captured only through finite fragments;

- In Kleene's number realizability:

  infinite sequences are given (wholly) by recursive indices.

Different Treatments of Reals

The contradiction seems to be on the difference:

- In Brouwer's idea:

  infinite sequences can be captured only through finite fragments;

- In Kleene's number realizability:

  infinite sequences are given (wholly) by recursive indices.

This contrast becomes clearer in

- $\forall\alpha$-case of BHK interpretation:
  given via finite fragments vs. given by an index;

Different Treatments of Reals

The contradiction seems to be on the difference:

- In Brouwer's idea:

  infinite sequences can be captured only through finite fragments;

- In Kleene's number realizability:

  infinite sequences are given (wholly) by recursive indices.

This contrast becomes clearer in

- $\forall\alpha$-case of BHK interpretation: given via finite fragments vs. given by an index;
- game semantics: Opponent's challenges can be seen via finite fragments vs. given by an index.

Our Motivating Questions

The actual content of Church's Thesis is

> "infinite sequences of numbers (or functions) are given by recursive indices"

rather than merely

> "any infinite sequence of numbers (or function) is recursive"

Our Motivating Questions

The actual content of Church's Thesis is

> "infinite sequences of numbers (or functions) are given by recursive indices"

rather than merely

> "any infinite sequence of numbers (or function) is recursive"

Questions

- How to formalize the latter statement?

Our Motivating Questions

The actual content of Church's Thesis is

> "infinite sequences of numbers (or functions) are given by recursive indices"

rather than merely

> "any infinite sequence of numbers (or function) is recursive"

Questions

- How to formalize the latter statement?
- Does it contradict BCP?

Our Motivating Questions

The actual content of Church's Thesis is

> "infinite sequences of numbers (or functions) are given by recursive indices"

rather than merely

> "any infinite sequence of numbers (or function) is recursive"

Questions

- How to formalize the latter statement?
- Does it contradict BCP?
- Can it play the role of Church's Thesis for Russian Recursive Constructive Mathematic?

Our Motivating Questions

The actual content of Church's Thesis is

> "infinite sequences of numbers (or functions) are given by recursive indices"

rather than merely

> "any infinite sequence of numbers (or function) is recursive"

Questions

- How to formalize the latter statement?
- Does it contradict BCP?
- Can it play the role of Church's Thesis for Russian Recursive Constructive Mathematic?
- Does it fit with Markov's original idea?

Negative Church's Thesis

Negative Church's Thesis NCT:

$$\forall\alpha\neg\forall e\neg\forall x(\{e\}(x)\downarrow\wedge\alpha(x)=\{e\}(x))$$

Negative Church's Thesis

Negative Church's Thesis NCT:

$$\forall\alpha\neg\forall e\neg\forall x(\{e\}(x){\downarrow}\wedge\alpha(x)=\{e\}(x))$$

Compared to the usual Church's Thesis CT:

$$\forall\alpha\exists e\forall x(\{e\}(x){\downarrow}\wedge\alpha(x)=\{e\}(x)).$$

Negative Church's Thesis

Negative Church's Thesis NCT:

$$\forall \alpha \neg \forall e \neg \forall x (\{e\}(x){\downarrow} \wedge \alpha(x) = \{e\}(x))$$

Compared to the usual Church's Thesis CT:

$$\forall \alpha \exists e \forall x (\{e\}(x){\downarrow} \wedge \alpha(x) = \{e\}(x)).$$

NCT is equivalent to CT^N over **HA** $+$ MP where

$\varphi^N :\equiv \neg\neg\varphi \quad (\varphi$ is atomic); $\qquad (\varphi \wedge \psi)^N :\equiv \varphi^N \wedge \psi^N;$

$(\varphi \vee \psi)^N :\equiv \neg(\neg\varphi^N \wedge \neg\psi^N); \quad (\varphi \to \psi)^N :\equiv \varphi^N \to \psi^N;$

$(\exists x \varphi[x])^N :\equiv \neg\forall x \neg(\varphi[x]^N); \quad (\forall x \varphi[x])^N :\equiv \forall x (\varphi[x]^N);$

$(\exists x \varphi[\alpha])^N :\equiv \neg\forall x \neg(\varphi[\alpha]^N); \quad (\forall x \varphi[\alpha])^N :\equiv \forall x (\varphi[\alpha]^N).$

Functional Algebra

Functionals can be coded by functions (by Brouwer?)

$(\alpha|\beta)(n) = \alpha(\langle n \rangle * \beta)$ where

$$\alpha(\beta) = \begin{cases} \alpha(\beta \restriction m) - 1 & \text{for } m = \min\{k \mid \alpha(\beta \restriction k) > 0\} \\ \text{undefined} & \text{if there is no such } k. \end{cases}$$

Functional Algebra

Functionals can be coded by functions (by Brouwer?)

$(\alpha|\beta)(n) = \alpha(\langle n \rangle * \beta)$ where

$$\alpha(\beta) = \begin{cases} \alpha(\beta \restriction m) - 1 & \text{for } m = \min\{k \mid \alpha(\beta \restriction k) > 0\} \\ \text{undefined} & \text{if there is no such } k. \end{cases}$$

Note: $\alpha(\beta) \in \omega$ and $\alpha|\beta \in \omega^\omega$ (if defined) for $\alpha, \beta \in \omega^\omega$.

Functional Algebra

Functionals can be coded by functions (by Brouwer?)

$(\alpha|\beta)(n) = \alpha(\langle n\rangle * \beta)$ where

$$\alpha(\beta) = \begin{cases} \alpha(\beta\restriction m) - 1 & \text{for } m = \min\{k \mid \alpha(\beta\restriction k) > 0\} \\ \text{undefined} & \text{if there is no such } k. \end{cases}$$

<u>Note:</u> $\alpha(\beta) \in \omega$ and $\alpha|\beta \in \omega^\omega$ (if defined) for $\alpha, \beta \in \omega^\omega$.

$(\omega^\omega, |\,)$ forms a Partial Combinator Algebra (PCA):

$$\exists\kappa(\kappa|\alpha|\beta \simeq \alpha); \qquad \exists\sigma(\sigma|\alpha|\beta|\gamma \simeq (\alpha|\gamma)|(\beta|\gamma)).$$

Functional Algebra

Functionals can be coded by functions (by Brouwer?)

$(\alpha|\beta)(n) = \alpha(\langle n\rangle * \beta)$ where

$$\alpha(\beta) = \begin{cases} \alpha(\beta\restriction m) - 1 & \text{for } m = \min\{k \mid \alpha(\beta\restriction k) > 0\} \\ \text{undefined} & \text{if there is no such } k. \end{cases}$$

Note: $\alpha(\beta) \in \omega$ and $\alpha|\beta \in \omega^\omega$ (if defined) for $\alpha, \beta \in \omega^\omega$.

$(\omega^\omega, |\,)$ forms a Partial Combinator Algebra (PCA):

$$\exists\kappa(\kappa|\alpha|\beta \simeq \alpha); \qquad \exists\sigma(\sigma|\alpha|\beta|\gamma \simeq (\alpha|\gamma)|(\beta|\gamma)).$$

This is called Kleene second model.

Functional Realizability

The realizability can be defined by this PCA:

$$\alpha\,\mathfrak{fr}\,\varphi \equiv \varphi \quad \text{for atomic } \varphi;$$

$$\alpha\,\mathfrak{fr}\,\varphi \wedge \psi \equiv ((\alpha)_0\,\mathfrak{fr}\,\varphi) \wedge ((\alpha)_1\,\mathfrak{fr}\,\psi);$$

$$\alpha\,\mathfrak{fr}\,\varphi \vee \psi \equiv (\alpha(0) = 0 \wedge (\alpha)_1\,\mathfrak{nr}\,\varphi) \vee (\alpha(0) \neq 0 \wedge (\alpha)_1\,\mathfrak{fr}\,\psi);$$

$$\alpha\,\mathfrak{fr}\,\varphi \to \psi \equiv \forall \beta((\beta\,\mathfrak{fr}\,\varphi) \to ((\alpha|\beta)\!\downarrow \wedge\, (\alpha|\beta)\,\mathfrak{fr}\,\psi));$$

$$\alpha\,\mathfrak{fr}\,\exists \xi \varphi[\xi] \equiv (\alpha)_1\,\mathfrak{fr}\,\varphi[(\alpha)_0];$$

$$\alpha\,\mathfrak{fr}\,\forall \xi \varphi[\xi] \equiv \forall \xi((\alpha|\xi)\!\downarrow \wedge\, (\alpha|\xi)\,\mathfrak{fr}\,\varphi[\xi]).$$

Functional Realizability

The realizability can be defined by this PCA:

$$\alpha \,\mathfrak{fr}\, \varphi \equiv \varphi \quad \text{for atomic } \varphi;$$

$$\alpha \,\mathfrak{fr}\, \varphi \wedge \psi \equiv ((\alpha)_0 \,\mathfrak{fr}\, \varphi) \wedge ((\alpha)_1 \,\mathfrak{fr}\, \psi);$$

$$\alpha \,\mathfrak{fr}\, \varphi \vee \psi \equiv (\alpha(0) = 0 \wedge (\alpha)_1 \,\mathfrak{nr}\, \varphi) \vee (\alpha(0) \neq 0 \wedge (\alpha)_1 \,\mathfrak{fr}\, \psi);$$

$$\alpha \,\mathfrak{fr}\, \varphi \to \psi \equiv \forall \beta ((\beta \,\mathfrak{fr}\, \varphi) \to ((\alpha|\beta)\downarrow \wedge (\alpha|\beta) \,\mathfrak{fr}\, \psi));$$

$$\alpha \,\mathfrak{fr}\, \exists \xi \varphi[\xi] \equiv (\alpha)_1 \,\mathfrak{fr}\, \varphi[(\alpha)_0];$$

$$\alpha \,\mathfrak{fr}\, \forall \xi \varphi[\xi] \equiv \forall \xi ((\alpha|\xi)\downarrow \wedge (\alpha|\xi) \,\mathfrak{fr}\, \varphi[\xi]).$$

If φ is <u>ess. $(\vee, \exists)$-free</u> (i.e., built by $\wedge, \to, \forall$ from Σ^0_1),

$$\varphi \leftrightarrow \exists \alpha (\alpha \,\mathfrak{fr}\, \varphi)$$

Functional Realizability

The realizability can be defined by this PCA:

$$\alpha\,\mathfrak{fr}\,\varphi \equiv \varphi \quad \text{for atomic } \varphi;$$

$$\alpha\,\mathfrak{fr}\,\varphi \wedge \psi \equiv ((\alpha)_0\,\mathfrak{fr}\,\varphi) \wedge ((\alpha)_1\,\mathfrak{fr}\,\psi);$$

$$\alpha\,\mathfrak{fr}\,\varphi \vee \psi \equiv (\alpha(0) = 0 \wedge (\alpha)_1\,\mathfrak{nr}\,\varphi) \vee (\alpha(0) \neq 0 \wedge (\alpha)_1\,\mathfrak{fr}\,\psi);$$

$$\alpha\,\mathfrak{fr}\,\varphi \to \psi \equiv \forall\beta((\beta\,\mathfrak{fr}\,\varphi) \to ((\alpha|\beta)\!\downarrow \wedge\, (\alpha|\beta)\,\mathfrak{fr}\,\psi));$$

$$\alpha\,\mathfrak{fr}\,\exists\xi\varphi[\xi] \equiv (\alpha)_1\,\mathfrak{fr}\,\varphi[(\alpha)_0];$$

$$\alpha\,\mathfrak{fr}\,\forall\xi\varphi[\xi] \equiv \forall\xi((\alpha|\xi)\!\downarrow \wedge\, (\alpha|\xi)\,\mathfrak{fr}\,\varphi[\xi]).$$

If φ is ess. $(\vee, \exists)$-free (i.e., built by $\wedge, \to, \forall$ from Σ^0_1),

$$\mathbf{EL}_0 \vdash \varphi \leftrightarrow \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi)$$

Elementary Arithmetic EL

iQex consists of:

$$x+0=x; \qquad x+(y+1)=(x+y)+1;$$
$$x\cdot 0=0; \qquad x\cdot(y+1)=(x\cdot y)+x;$$
$$\exp(x,0)=1; \qquad \exp(x,y+1)=\exp(x,y)\cdot x;$$
$$\neg(x<x); \qquad x<y\wedge y<z\to x<z;$$
$$x<x+1; \qquad x<y\to(x+1<y)\vee(x+1=y).$$

$\mathbf{EL}_0$ extends **iQex** by

- $\alpha\restriction 0=\langle\,\rangle$; and $\alpha\restriction(x+1)=(\alpha\restriction x)*\langle\alpha(x)\rangle$;
- Induction for Σ^0_1 formulae;
- Δ^0_0 bounded search: for Δ^0_0 formula φ
  $\exists\alpha\forall x((\exists y<t[x])\varphi[x,y]\to\alpha(x)<t[x]\wedge\varphi[x,\alpha(x)])$.

Elementary Arithmetic EL

iQex consists of:

$$x+0=x; \qquad x+(y+1)=(x+y)+1;$$
$$x\cdot 0=0; \qquad x\cdot(y+1)=(x\cdot y)+x;$$
$$\exp(x,0)=1; \qquad \exp(x,y+1)=\exp(x,y)\cdot x;$$
$$\neg(x<x); \qquad x<y\wedge y<z\to x<z;$$
$$x<x+1; \qquad x<y\to(x+1<y)\vee(x+1=y).$$

$\mathbf{EL}_0$ extends **iQex** by

- $\alpha\restriction 0=\langle\,\rangle$; and $\alpha\restriction(x+1)=(\alpha\restriction x)*\langle\alpha(x)\rangle$;
- Induction for Σ^0_1 formulae;
- Δ^0_0 bounded search: for Δ^0_0 formula φ $\exists\alpha\forall x((\exists y<t[x])\varphi[x,y]\to\alpha(x)<t[x]\wedge\varphi[x,\alpha(x)])$.

Note: $\mathbf{EL}_0$ is consistent with CT.

Realizability Model

It is known that

$$\mathbf{EL}_0 + \mathsf{BCP} \vdash \varphi \implies \mathbf{EL}_0 \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi).$$

Actually, usually this is used in the form:

$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{WFT} \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{WKL} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi);$$
$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{FT} \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{KL} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi);$$

Realizability Model

It is known that

$$\mathbf{EL}_0 + \mathsf{BCP} \vdash \varphi \implies \mathbf{EL}_0 \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi).$$

Actually, usually this is used in the form:

$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{WFT} \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{WKL} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi);$$
$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{FT} \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{KL} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi);$$
$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{BI}_M \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{BI} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi).$$

Realizability Model

It is known that

$$\mathbf{EL}_0 + \mathsf{BCP} \vdash \varphi \implies \mathbf{EL}_0 \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi).$$

Actually, usually this is used in the form:

$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{WFT} \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{WKL} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi);$$

$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{FT} \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{KL} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi);$$

$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{BI}_M \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{BI} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi).$$

We can use the same technique to show

$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{NCT} \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{CT} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi).$$

Realizability Model

It is known that

$$\mathbf{EL}_0 + \mathsf{BCP} \vdash \varphi \implies \mathbf{EL}_0 \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi).$$

Actually, usually this is used in the form:

$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{WFT} \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{WKL} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi);$$

$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{FT} \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{KL} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi);$$

$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{BI}_M \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{BI} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi).$$

We can use the same technique to show

$$\mathbf{EL}_0 + \mathsf{BCP} + \mathsf{NCT} \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{CT} \vdash \exists\alpha(\alpha\,\mathfrak{fr}\,\varphi).$$

<u>Conclusion:</u> NCT is consistent with BCP.

Classical Interpreting Theories

In classical theories, we can realize MP:

$$\mathbf{EL}_0 + \mathsf{MP} + \mathsf{BCP} + \mathsf{NCT} \vdash \varphi$$
$$\Longrightarrow \mathbf{EL}_0 + \mathsf{LEM} + \mathsf{CT} \vdash \exists\alpha(\alpha\, \mathfrak{fr}\, \varphi).$$

Classical Interpreting Theories

In classical theories, we can realize MP:

$$\mathbf{EL}_0 + \mathsf{MP} + \mathsf{BCP} + \mathsf{NCT} \vdash \varphi$$
$$\Longrightarrow \; \mathbf{EL}_0 + \mathsf{LEM} + \mathsf{CT} \vdash \exists \alpha (\alpha \, \mathfrak{fr} \, \varphi).$$

Question for Russian Constructivists:

Can the functional realizability model $\mathfrak{fr}$ (defined inside $\mathbf{EL}_0 + \mathsf{LEM} + \mathsf{CT}$) be considered as a model of Russian Recursive Constructive Mathematics?

Classical Interpreting Theories

In classical theories, we can realize MP:

$$\mathbf{EL}_0 + \mathsf{MP} + \mathsf{BCP} + \mathsf{NCT} \vdash \varphi \implies \mathbf{EL}_0 + \mathsf{LEM} + \mathsf{CT} \vdash \exists\alpha(\alpha\, \mathfrak{fr}\, \varphi).$$

Question for Russian Constructivists:

Can the functional realizability model $\mathfrak{fr}$ (defined inside $\mathbf{EL}_0 + \mathsf{LEM} + \mathsf{CT}$) be considered as a model of Russian Recursive Constructive Mathematics?

- It satisfies Markov's Principle;
- It satisfies the Axiom of Choice as well as our formulation of "every function is recursive";
- It satisfies also Brouwer's Continuity Principle.

KLS Theorem

One of the most important theorems in RRCM is

Any function $f : \mathbb{R} \to \mathbb{R}$ is continuous

where $\mathbb{R}$ is the real numbers (defined constructively).

KLS Theorem

One of the most important theorems in RRCM is

Any function $f : \mathbb{R} \to \mathbb{R}$ is continuous

where $\mathbb{R}$ is the real numbers (defined constructively).

- The proof in $\mathbf{EL}_0 + \mathsf{MP} + \mathsf{ECT}$ is involved (due to Kreisel-Lacombe-Shoenfield);
- That in $\mathbf{EL}_0 + \mathsf{BCP}$ is straightforward.

KLS Theorem

One of the most important theorems in RRCM is

Any function $f : \mathbb{R} \to \mathbb{R}$ is continuous

where $\mathbb{R}$ is the real numbers (defined constructively).

- The proof in $\mathbf{EL}_0 + \mathsf{MP} + \mathsf{ECT}$ is involved (due to Kreisel-Lacombe-Shoenfield);
- That in $\mathbf{EL}_0 + \mathsf{BCP}$ is straightforward.

Question for Russian Constructivists 2:

Are there other important theorems in RRCM?

Early Intuitionism?

In early stages, Brouwer rejected lawless sequences:

- lawful sequence “=” recursive function?
- lawless sequence “=” non-recursive one?

Early Intuitionism?

In early stages, Brouwer rejected lawless sequences:

- lawful sequence "=" recursive function?
- lawless sequence "=" non-recursive one?

He changed his mind; indeed Kleene found

> a recursive finitary branching tree with no recursive infinite path nor finite height;

particularly Brouwer's FT contradicts CT.

Early Intuitionism?

In early stages, Brouwer rejected lawless sequences:

- lawful sequence "=" recursive function?
- lawless sequence "=" non-recursive one?

He changed his mind; indeed Kleene found

> a recursive finitary branching tree with no recursive infinite path nor finite height;

particularly Brouwer's FT contradicts CT.

Question for Historians of Brouwer

Can our model $\mathfrak{fr}$ (defined in $\mathbf{EL}_0 + \mathsf{LEM} + \mathsf{CT}$) be seen as a model of this early stage of Intuitionism?

- it satisfies "every sequence is recursive";
- it still satisfies BCP.

Kleene's Alternative

Veldmann (2014) introduced Kleene's Alternative KA:

> there is a finitary branching tree with no infinite path nor finite height.

Kleene's Alternative

Veldmann (2014) introduced Kleene's Alternative KA:

> there is a finitary branching tree with no infinite path nor finite height.

This can be analyzed as

- $\mathrm{Bar}[T] :\equiv \mathrm{FinTr}[T] \wedge \neg\exists\alpha \mathrm{Path}[\alpha, T]$;
- KA is $\exists T(\mathrm{Bar}[T] \wedge \neg\exists n \mathrm{Ht}[n, T])$;
- whereas FT is $\forall T(\mathrm{Bar}[T] \rightarrow \exists n \mathrm{Ht}[n, T])$.

Kleene's Alternative

Veldmann (2014) introduced Kleene's Alternative KA:

> there is a finitary branching tree with no infinite path nor finite height.

This can be analyzed as

- $\mathrm{Bar}[T] :\equiv \mathrm{FinTr}[T] \wedge \neg\exists\alpha \mathrm{Path}[\alpha, T]$;
- KA is $\exists T(\mathrm{Bar}[T] \wedge \neg\exists n \mathrm{Ht}[n, T])$;
- whereas FT is $\forall T(\mathrm{Bar}[T] \to \exists n \mathrm{Ht}[n, T])$.

Veldman realized that

- Many equivalents of FT (e.g., int-value theorem) are of the form $\forall\alpha(\varphi[\alpha] \to \exists\beta\psi[\alpha, \beta])$ and
- $\exists\alpha(\varphi[\alpha] \wedge \neg\exists\beta\psi[\alpha, \beta])$'s are equivalent to KA.

Kleene's Alternative

Veldmann (2014) introduced Kleene's Alternative KA:

> there is a finitary branching tree with no infinite path nor finite height.

This can be analyzed as

- $\mathrm{Bar}[T] :\equiv \mathrm{FinTr}[T] \wedge \neg\exists\alpha \mathrm{Path}[\alpha, T]$;
- KA is $\exists T(\mathrm{Bar}[T] \wedge \neg\exists n \mathrm{Ht}[n, T])$;
- whereas FT is $\forall T(\mathrm{Bar}[T] \rightarrow \exists n \mathrm{Ht}[n, T])$.

Veldman realized that

- Many equivalents of FT (e.g., int-value theorem) are of the form $\forall\alpha(\varphi[\alpha] \rightarrow \exists\beta\psi[\alpha, \beta])$ and
- $\exists\alpha(\varphi[\alpha] \wedge \neg\exists\beta\psi[\alpha, \beta])$'s are equivalent to KA.

Now $\mathbf{EL}_0 + \mathsf{MP} + \mathsf{NCT} \vdash \mathsf{KA}$.

Summary

1. A new axiom NCT is introduced which asserts
   "every infinite sequence is recursive"
   without requiring
   "a infinite sequence is given by an index".

Summary

1. A new axiom NCT is introduced which asserts

   “every infinite sequence is recursive”

   without requiring

   “a infinite sequence is given by an index”.

2. NCT is consistent with BCP which requires

   “infinite sequences can be accessed only via finite fragments”.

Summary

1. A new axiom NCT is introduced which asserts

   "every infinite sequence is recursive"

   without requiring

   "a infinite sequence is given by an index".
2. NCT is consistent with BCP which requires

   "infinite sequences can be accessed only via finite fragments".
3. There is a model of NCT + MP which satisfies (one of) the most important theorem of RRCM:

   "any function $f : \mathbb{R} \to \mathbb{R}$ is continuous".

Summary

1. A new axiom NCT is introduced which asserts

   "every infinite sequence is recursive"

   without requiring

   "a infinite sequence is given by an index".
2. NCT is consistent with BCP which requires

   "infinite sequences can be accessed only via finite fragments".
3. There is a model of NCT + MP which satisfies (one of) the most important theorem of RRCM:

   "any function $f : \mathbb{R} \to \mathbb{R}$ is continuous".
4. We already knew many mathematical consequences of NCT + MP via Veldman's KA.

Summary

1. A new axiom NCT is introduced which asserts

   "every infinite sequence is recursive"

   without requiring

   "a infinite sequence is given by an index".
2. NCT is consistent with BCP which requires

   "infinite sequences can be accessed only via finite fragments".
3. There is a model of NCT + MP which satisfies (one of) the most important theorem of RRCM:

   "any function $f : \mathbb{R} \to \mathbb{R}$ is continuous".
4. We already knew many mathematical consequences of NCT + MP via Veldman's KA.
5. This might model Brouwer's idea before the acceptance of lawless sequence.

References

The technical results in the talk will be published in

- T. Nemoto and K. Sato
  "A marriage of Brouwer's Intuitionism and Hilbert's Finitism I: Arithmetic",
  to appear in *The Journal of Symbolic Logic*.

References

The technical results in the talk will be published in

- T. Nemoto and K. Sato
  "A marriage of Brouwer's Intuitionism and Hilbert's Finitism I: Arithmetic",
  to appear in *The Journal of Symbolic Logic*.

Other references:

- A. Troelstra and D. van Dalen,
  "Constructivism in Mathematics" Volumes I,
  Elsevier, 1988.
- W. Veldman,
  "Brouwer's fan theorem as an axiom and as a contrast to Kleene's alternative",
  Archive for Mathematical Logic 53 pp.621-693, 2014.