Do we really need Ex Falso?

A first reconnaissance

Dick de Jongh[1]

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Steklov Mathematical Institute, Moscow, Russia

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[1] In cooperation with Noor Heerkens

Introduction

By minimal propositional logic, MPC, we think of MPC_f,

$\mathcal{L}_{\mathsf{MPC}} = \{\wedge, \vee, \rightarrow, f\}$, a constant f representing *falsum*.

We are interested in differences between minimal and intuitionistic logic; does minimal logic allow us to do everything we can do with intuitionistic logic? And does it have anything to do with negationless mathematics? We look at

1. Heyting Arithmetic HA and intuitionistic analysis,
2. The logic of equality and apartness
3. second-order Heyting arithmetic.

So far, we have not discovered profound differences.
But the problem is less clear than we first thought.

We thank Joan Rand Moschovakis and Anne Troelstra for suggestions.

1=0 AS $\bot$

We consider Elementary Analysis EL.

Proposition $\bot$ is definable in EL as $0 = S0$.

Proof. First we
show that $\vdash_{\mathsf{EL}} 0 = S0 \to \forall x(x = 0)$:

$$x = 0 : \text{Since } \vdash_{\mathsf{EL}} 0 = 0, \text{ also } \vdash_{\mathsf{EL}} 0 = S0 \to 0 = 0;$$

$$x \to Sx : \ \vdash_{\mathsf{EL}} 0 = S0 \to x = 0 \implies \vdash_{\mathsf{EL}} 0 = S0 \to Sx = S0$$

$$\text{So, } \vdash_{\mathsf{EL}} 0 = S0 \to Sx = 0.$$

Thus $0 = S0 \to (x = 0 \wedge y = 0) \to x = y$.

Atomic formulas of EL are of the form $s = t$.

Since $\vdash_{\mathsf{EL}} 0 = S0 \to \forall xy(x = y)$, $\vdash_{\mathsf{EL}} 0 = S0 \to s = t$ for all s, t.

By formula induction, $\vdash_{\mathsf{EL}} 0 = S0 \to A$ for all formulas A.

What is the extension of this result?

Clearly this result extends to any system where function symbols are used exclusively by way of their number-theoretic values.

Even equality between functions can be added as long as this equality is extensional.

WHY $1=0$ AS f?

In the first place this result holds for HA. How reasonable Is it to take $1=0$ for f?

The only negation that occurs in the axioms of HA is in $\forall x \neg (Sx = 0)$, which we can rewrite as $\neg \exists x (Sx = 0)$, i.e. $\exists x (Sx = 0) \rightarrow f$.

Looking at it this way it seems one might take $\exists x (Sx = 0)$ as f. But it is not difficult to see that in HA without this axiom one can prove $1=0$ from $\exists x (Sx = 0)$ so that makes no difference.

More on $1=0$ as f

There are general phenomena here. Let T be a theory over IQM composed of a positive (i.e. without negations) part T^+ plus $\neg T_0$ with T_0 positive. This is like the case of HA. It seems a reasonable choice to take T_0 as f.

Now, if $T \vdash \neg A$, i.e. $T^+ \vdash \neg T_0 \to \neg A$, then $T^+ \vdash A \to T_0$:

We have $T^+ \vdash (T_0 \to f) \to (A \to f)$. Just substitute T_0 for f which is just a variable.

So, if T_0 proves all sentences, or a subclass of them (as we will encounter further on) then so does A.

Logic of Equality and Apartness

Over IQC the Theory of Apartness, AP, has the axioms of EQ, the theory of identity plus:

$$(1)\ \forall xyz(x = y \to (x \# z \to y \# z))$$
$$(2)\ \forall xy(\neg(x \# y) \leftrightarrow x = y)$$
$$(3)\ \forall xy(x \# y \to y \# x)$$
$$(4)\ \forall xyz(x \# y \to z \# x \vee z \# y)$$

Note that if an apartness relation exists then $\forall xy(x = y \to \neg\neg(x = y))$ (= is stable).

Decidable equality

In intuitionistic logic we know that if equality is decidable on a set, i.e. $\forall xy(x = y \vee x \neq y)$, then $\neq$ is an apartness relation. This needs:

$$\vdash_{\mathsf{IPC}} (p \vee \neg p) \to (\neg\neg p \to p),$$

which is an instance of the general scheme of the disjunctive syllogism:

$$\vdash_{\mathsf{IPC}} (p \vee q) \to (\neg p \to q).$$

The disjunctive syllogism is equivalent to MPC over IPC, and this consequence is not valid.

But it is open to find a nice example of a set with a decidable but non-stable equality over MQC.

The equality theory of apartness in IQC

Over IQC, $\forall z(z \neq x \vee z \neq y)$ is a stronger relation than $x \neq y$ (substitute x for z).
We can iterate this by defining the following inequalities:

$$x \neq_0 y := x \neq y)$$
$$x \neq_{n+1} y := \forall z(z \neq_n x \vee z \neq_n y)$$

We define the following generalized stability axioms:

$$S_n := \forall xy(\neg(x \neq_n y) \to x = y).$$

and

$$\mathsf{SEQ}^\omega := \mathsf{EQ} + \{\mathsf{S}_n \mid n \in \omega\}.$$

Van Dalen and Statman proved that AP is conservative over SEQ^ω, i.e. SEQ^ω is the equality fragment of AP.

THE EQUALITY THEORY OF APARTNESS IN MQC

Over MQC, $x \neq_1 y : \ \forall z(z \neq x \vee z \neq y)$ is no longer a stronger relation than $x \neq y$.
Again by the failure of the disjunctive syllogism. This iterates.

The principles you need to get it going, $\mathsf{I}_{m,n}$, are provable in EQ over IQC, but not over MQC:

$$x \neq_n y \to x \neq_m y \ \text{ for } n \geq m \qquad (\mathsf{I}_{n,m})$$

If we add $\mathsf{I}_{n,m}$ for $n \geq m$ to SEQ^ω to obtain SEQM^ω we can prove van Dalen-Statman for MPC:

Over MQC the equality fragment of AP is SEQM^ω.

Instead of the proof-theoretic proof by van Dalen-Statman (later simplified by Troelstra) we use the more straightforward semantic proof by Smoryński.

The difference between IQC and MQC is rather marginal here.

Second-order Heyting Arithmetic

By second-order Heyting arithmetic, HAS, we understand the intuitionistic theory of species, given by HA together with unary species variables $X_0, X_1, \ldots$ and certain axioms, including full comprehension.
Xx will stand for $x \in X$.

deJSmorynski(1976) showed the consistency of the principles AC-NS, UP!, UP^c, MP and IP_0 with HAS plus the contradictoriness of non-intuitionistic logical principles.

We replicate, for MQC, some but not all of these results.

Frames for second-order Heyting Arithmetic

An HAS frame for minimal logic is $(K, \leq, F, D_1, D_2)$ where $(K, \leq)$ is a po set with root α_0, $F \subseteq K$ is $\leq$-upwards closed.

D_1 and D_2 are domain functions.

$\alpha \in K \Longrightarrow D_1\alpha = \omega = \{0, 1, \ldots\}$

$D_2\alpha$ is the family of all systems of sets, or species.

A species T is a set $\{T_\alpha \mid \alpha \in K\}$ with $T_\alpha \subseteq \omega$, such that:

$$\alpha \leq \beta \Rightarrow T_\alpha \subseteq T_\beta$$

Models for second-order Heyting Arithmetic

An HAS model $\mathcal{M}$ for minimal logic is $(K, \leq, F, D_1, D_2, \Vdash)$ where $(K, \leq, F, D_1, D_2)$ is an HAS frame for MPC and $\Vdash$ a forcing relation:

$$\alpha \Vdash A \quad \Leftrightarrow \quad \omega \Vdash A \text{ (for } A \text{ atomic in } \mathscr{L}(HA))$$
$$\alpha \Vdash f \quad \Leftrightarrow \quad \alpha \in F$$
$$\alpha \Vdash t \in T \quad \Leftrightarrow \quad t \in T_\alpha \text{ (where } t \in \omega \text{ and } T = \{T_\alpha \ : \ \alpha \in K\})$$

$$\alpha \Vdash \exists X A(X) \quad \Leftrightarrow \quad \text{for some } T \in D_2\alpha : \ \alpha \Vdash A(T)$$
$$\alpha \Vdash \forall X A(X) \quad \Leftrightarrow \quad \text{for all } \beta \geq \alpha \text{ and all } T \in D_2\beta : \ \beta \Vdash A(T)$$

Connectives and number quantifiers like usual.
Equality, =, is in the models just an equivalence (and congruence) relation.

We start by taking $\mathfrak{M}$ with the $\aleph_0$-ary infinite tree as $(K, \leq)$.
That $\mathfrak{M}$ satisfies HAS is easy in the case of both IQC and MQC.

1=0 as Ex Falso

Again we take f to be $0{=}1$. Again, $\alpha \Vdash 0{=}1 \implies \alpha \Vdash \forall xy(x = y)$.

So, $\forall T \in D_2, \alpha(T_\alpha = \emptyset \vee T_\alpha = \omega)$

and, if $\alpha \Vdash 0{=}1$, then α, $\alpha \Vdash f \to \forall X(\exists xXx \to \forall xXx)$.

Although HAS is not like HA the same in minimal and intuitionistic logic if we interpret f as $0{=}1$, its influence is still heavily felt.

Of course, we keep $\vdash_{\mathsf{HA}} 0{=}1 \to A$ for all purely arithmetical A.

Uniformity principle, and falsifying unprovable formulas

Let $\mathcal{M}$ be an HAS model for minimal logic on the full $\aleph_0$-ary tree, then $\mathfrak{M}$ satifsies the Weak Uniformity Principle:

$$\forall X \exists ! y\, A(X,y) \to \exists ! y \forall X\, A(X,y). \tag{UP!}$$

Also establishment of $\mathsf{AC}_{\mathsf{NS}}$ in the model $\mathfrak{M}$ is unproblematic.

Let $A(p_1,\ldots,p_n,f)$ be an unprovable formula of MPC.
Then $\mathcal{M}$ satisfies $\mathcal{M} \not\models A(\exists x X_1 x, \ldots, \exists x X_n x, 0{=}1)$
for some species $X_1,\ldots,X_n$.

Proof of $\not\models A$

Suppose $\nvdash_{\mathsf{MPC}} A(p_1, \ldots, p_n, f)$. There exists a finite Kripke countermodel for A. Since the full $\aleph_0$-ary tree $(K, \leq)$ can be projected on this finite model through a p-morphism, we find a countermodel $\mathcal{M}$ on $(K, \leq)$ such that $\mathcal{M} \not\models A(p_1, \ldots, p_n, f)$.

We define species $X_i = \{X_i, \alpha \mid \alpha \in K\}$, where:

$$X_i, \alpha = \begin{cases} \emptyset & \text{if } \alpha \nVdash p_i \\ \omega & \text{if } \alpha \Vdash p_i \end{cases}$$

Furthermore we define:

$$\alpha \Vdash 0{=}1 \;\Leftrightarrow\; \alpha \Vdash f$$

By definition of the species we then obtain
$\mathcal{M} \not\models A(\exists x X_1 x, \ldots, \exists x X_n x, 0{=}1)$.

A stronger form of falsifying unprovable formulas

dJSm obtained for IPC the much stronger result that for all unprovable A, $\mathfrak{M} \models \neg\forall X_1 \ldots X_n A(\exists x X_1 x, \ldots, \exists x X_n x)$, even for first order logic which we did not yet consider.

To obtain similar results for MQC it would be necessary to construct models that even with respect to the regions where f holds are much more homogeneous than the models so far. We are using models built on $\mathbb{Q}$ instead of $\mathbb{N}$.

These models also work for the general principle UP without parameters.

Markov's principle

The following principle does not hold in all such HAS models for minimal logic:

$$\forall X(\forall x((Xx \vee \neg Xx) \wedge \neg\neg\exists x Xx \to \exists x Xx)) \tag{MP}$$

Consider the HAS model on the full $\aleph_0$-ary tree $(K, \leq)$ in which the root α_0 has some successor β that forces f.

Define the species T by $T_\alpha = \emptyset$ for all $\alpha \in K$.

Since $T \in D_2\beta$ and $\beta \Vdash \forall x(Tx \vee (Tx \to f)) \wedge ((\exists x Tx \to f) \to f)$, since f is true in β, yet $\beta \nVdash \exists x Tx$, we conclude $\alpha_0 \nVdash$ MP.

Whether we can apply the methods of deJSm to Markov's principle in MPC seems highly questionable.

Why $0=1$ as f?

Anne Troelstra suggested using $\forall X_0(X_0)$ as interpretation of f where X_0 runs over propositions:
$\forall X_0 = \forall X((\exists x(x \in X) \leftrightarrow \forall x(x \in X)) \to \ldots$.
If one does that, obviously $f \to A$ for all A is derivable in HAS over IQM, and as for HA there is no difference between minimal and intuitionistic logic. This shows that conceptual choices have to be made.
One can simply reject this choice for f. Another option is to consider $\mathsf{HAS}_{\mathsf{arith}}$ with only arithmetic comprehension. That is formally right but also awkward.

One might even go so far as to wonder: isn't it better to simply look at negationless mathematics again?

THANKS

Questions or Remarks?!