How Widespread Are Justification Logics?

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The Workshop on Proof Theory, Modal Logic
and Reflection Principles
Moscow 2017

You probably know something
about justification logics already.

You certainly know lots
about modal logics.

So my background will be rather casual.

Modal logics have $\Box X$,
(X is necessary).

Justification logics have $t{:}X$,
(X is so for reason t).

What it means to be a reason
varies from logic to logic.

It can be quite arbitrary.

Just as arbitrary as what *necessity*
means in modal logics.

Justification Terms

These are built up from *justification variables*

(which represent arbitrary justifications.)

And *justification constants*

(which justify unanalyzed, accepted truths: axioms.)

Building up uses *justification function symbols*.

A minimum set consists of $\cdot$ and $+$.

$\cdot$ embodies *modus ponens*:
if t justifies $X \supset Y$ and u justifies X
then $t \cdot u$ is intended to justify Y.

$+$ embodies *weakening*:
if either t or u justifies X
then $t + u$ also justifies X.

There may be other function symbols,
depending on the particular logic.

The Logic J_0

Axiom schemes:

All tautologies (or enough of them).

$t{:}(X \supset Y) \supset [u{:}X \supset (t \cdot u){:}y]$

$[t{:}X \vee u{:}X] \supset (t + u){:}X$

Rule:

$X, X \supset Y \Rightarrow Y$ (*modus ponens*)

Constant Specification

I'm skipping details here.

A constant specification $\mathcal{CS}$ assigns some constant to each axiom, and to each member of $\mathcal{CS}$.

J is J_0 plus some arbitrary constant specification.

Internalization

If X is provable in J
then $t{:}X$ is provable
for some justification term t.

Example – Proof in J

1. $X \supset (X \vee Y)$ (tautology)
2. $a{:}(X \supset (X \vee Y))$ (internalization)
3. $a{:}(X \supset (X \vee Y)) \supset (v_1{:}X \supset (a \cdot v_1){:}(X \vee Y))$ (axiom)
4. $v_1{:}X \supset (a \cdot v_1){:}(X \vee Y)$ (modus ponens)
5. $v_2{:}Y \supset (b \cdot v_1){:}(X \vee Y)$ (similarly)
6. $(a \cdot v_1){:}(X \vee Y) \supset (a \cdot v_1 + b \cdot v_2){:}(X \vee Y)$ (axiom)
7. $(b \cdot v_2){:}(X \vee Y) \supset (a \cdot v_1 + b \cdot v_2){:}(X \vee Y)$ (axiom)
8. $(v_1{:}X \vee v_2{:}Y) \supset (a \cdot v_1 + b \cdot v_2){:}(X \vee Y)$ (4, 5, 6, 7)

The Forgetful Functor

Define a map from justification formulas to modal formulas:

$$[t{:}X]^\circ = \Box X^\circ$$

and otherwise it's a boolean homomorphism.

This maps J theorems to K theorems.

True for axioms,
true for members of constant specification,
preserved by *modus ponens*.

For example,

$$[(v_1{:}X \vee v_2{:}Y) \supset (a \cdot v_1 + b \cdot v_2){:}(X \vee Y)]^\circ =$$
$$= (\Box P \vee \Box Q) \supset \Box(P \vee Q).$$

The central fact is that
this maps the theorems of J
onto the theorems of K.

Put another way,
every theorem of modal K
has an analysis of its reasoning in J.

Justification logic J and modal logic K are *counterparts*.

1. If X is provable in J using any constant specification then X° is a theorem of K.

2. If Y is a theorem of K then there is some justification formula X so that $X^\circ = Y$, where X is provable in J using some axiomatically appropriate constant specification.

Realization

In fact, X can have distinct justification variables where Y has negative necessitation operators. There is a hidden input/output modal structure.

Extending This Beyond J

There are infinitely many modal logics.

Given a modal logic,
does there exist a justification logic
that is a counterpart?

Which modal logics have
the kind of analysis a
justification logic provides?

One can add additional justification function symbols.

One can add additional axioms
governing those function symbols.

(And extend constant specifications
to cover these new axioms.)

One might even add additional rules of inference,
though this tends to be hard since
it can lead to problems
with *Internalization*.

The first justification logic was called LP,
for *logic of proofs*.

It was created by Sergei Artemov,
as an essential part of a project
to provide an arithmetic interpretation
of intuitionistic logic.

It adds a *justification checker* operation to J, denoted !

And an axiom scheme

$$t{:}X \supset !t{:}t{:}X$$

It also adds a *factivity* axiom scheme

$$t{:}X \supset X$$

LP corresponds to modal *S4*.

Without factivity, correspondence is with *K4*.

But, how far does this phenomenon go?

Which modal logics have
corresponding justification logics?

The heart of it is:
for what modal/justification pairs
is a *realization theorem* provable?

Realization
How Proved?

There are two kinds of realization proofs:
constructive
non-constructive

Constructive proofs are the most informative.

Non-constructive cover the broadest range.

Constructive proofs need a
cut-free modal proof procedure.
They use a *proof* as input.

So far the following have been used:
sequent calculi
semantic tableaus
hypersequents
nested sequents
prefixed tableaus

It is not known how to use
labeled deductive systems,
which would be the broadest machinery.

Non-constructive proofs use a possible world semantics for justification logics.

More about this shortly.

I will discuss some representative examples next.

The Original Example

Justification logic *LP*
corresponds to modal logic *S4*.

This is the oldest example.
For it, realization has been proved
by every known method.

S4 was chosen because
Gödel's translation mapped
intuitionistic logic to *S4*.

We use the version of the translation
that inserts a necessity symbol
in front of every intuitionistic subformula.

Well, what about other intermediate logics?

At the moment,
I only know how to handle two intermediate logics,
but this is already illustrative.

Weak Excluded Middle

Add to intuitionistic logic the scheme
$\neg X \vee \neg\neg X$

Also known as KC or Jankov's logic.

The smallest modal companion for this is $S4.2$

$S4$ plus $\Diamond\Box X \supset \Box\Diamond X$

Semantically, modal models are
S4 models that are *convergent*.

A justification counterpart called $JT4.2$ is
LP plus the following axiom scheme:

$$f(t,u){:}\neg t{:}X \vee g(t,u){:}\neg u{:}\neg X$$

Very Informal Idea

Using factivity, $(t{:}X \wedge u{:}\neg X) \supset \bot$

So $\neg t{:}X \vee \neg u{:}\neg X$ is provable.

In any context we have one of $\neg t{:}X$ or $\neg u{:}\neg X$.

$$f(t,u){:}\neg t{:}X \vee g(t,u){:}\neg u{:}\neg X$$

says we have a reason for the one that holds

An Example

$$[\Diamond\Box A \wedge \Diamond\Box B] \supset \Diamond\Box(A \wedge B)$$

is a theorem of $S4.2$

We rewrite it as

$$[\neg\Box\neg\Box A \wedge \neg\Box\neg\Box B] \supset \neg\Box\neg\Box(A \wedge B)$$

negative occurrences must become variables

A realization of it, provable in *JT4.2* is

$$\{\neg[j_4 \cdot j_3 \cdot !v_5 \cdot g(!v_3, j_2 \cdot v_5 \cdot !v_9)]{:}\neg v_9{:}A \wedge$$
$$\neg[j_5 \cdot f(!v_3, j_2 \cdot v_5 \cdot !v_9)]{:}\neg v_3{:}B\} \supset \neg v_5{:}\neg[j_1 \cdot v_9 \cdot v_3]{:}(A \wedge B)$$

j_1, j_2, j_3, j_4, j_5 are given using Internalization.

For instance j_1 internalizes a proof of

$$A \supset (B \supset (A \wedge B))$$

and j_2 internalizes a proof of

$$\neg[j_1 \cdot v_1 \cdot v_3]{:}(A \wedge B) \supset (v_1{:}A \supset \neg v_3{:}B)$$

No constructive proof of realization is known for S4.2.

(Maybe there is a constructive proof. Details need checking.)

Non-constructively we have the following very general result.

A *Geach logic* is one axiomatized over K using schemas of the form

$$\Diamond^k \Box^l X \supset \Box^m \Diamond^n X$$

All Geach logics have justification counterparts, connected via a Realization theorem.

In particular, infinitely many modal logics have justification counterparts.

In particular, *S4.2*.

Gödel-Dummet Logic

Add to intuitionistic logic the scheme

$$(X \supset Y) \vee (Y \supset X)$$

Also known as LC.

The smallest modal companion for this is $S4.3$

$S4$ plus $\Box(\Box X \supset Y) \vee \Box(\Box Y \supset X)$

Semantically, modal models are
S4 models that are *linear*.

A justification counterpart called $J4.3$ is LP plus the following axiom scheme:

$$f(t,u){:}(t{:}X \supset Y) \vee g(t,u){:}(u{:}Y \supset X)$$

Again, no constructive proof of realization is known.

And *S4.3* is not a Geach logic.

But we do have a very general semantic result that applies.

Earlier I mentioned semantics
for justification logics.

There are several.
The one we need now has become
known as *Fitting semantics*.

It is a possible world semantics.

$$\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$$

model

possible worlds

accessibility

atomic truth assignment

evidence function

$\mathcal{E}$ maps justification terms and formulas
to sets of possible worlds

Intuitively $\Gamma \in \mathcal{E}(t, X)$ says,
at possible world Γ,
t is *relevant* evidence for X.

This is the new thing,
added to the Kripke machinery.

$\mathcal{M}, \Gamma \Vdash t{:}X$ if and only if

Modal Condition: $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$

Evidence Condition: $\Gamma \in \mathcal{E}(t, X)$

X is 'necessary' at Γ and
t is relevant evidence for X at Γ

LP as an example

$$\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$$

$\langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ is an $S4$ model

$\cdot$ **Condition:** $\mathcal{E}(s, X \supset Y) \cap \mathcal{E}(t, X) \subseteq \mathcal{E}(s \cdot t, Y)$

$+$ **Condition:** $\mathcal{E}(s, X) \cup \mathcal{E}(t, X) \subseteq \mathcal{E}(s + t, X)$

Monotonicity Condition: If $\Gamma \mathcal{R} \Delta$ and $\Gamma \in \mathcal{E}(t, X)$ then $\Delta \in \mathcal{E}(t, X)$.

! **Condition:** $\mathcal{E}(t, X) \subseteq \mathcal{E}(!t, t{:}X)$.

Theorem: if KL is a canonical modal logic and JL is a candidate for a justification counterpart and the canonical justification model for JL is built on a KL frame, then KL realizes into JL.

This is how the result
mentioned earlier
about Geach logics is proved.

It also works for *S4.3* and *J4.3*,
though *S4.3* is not Geach.

A Different Kind of Example

I recently found a very simple justification counterpart for Gödel-Löb logic.

It isn't the first version of something like this. Daniyar Shamkanov has a different such logic. I don't know the relationships between them yet.

And also rather unexpectedly,
here realization has a constructive proof,
but I don't know how to
prove it semantically.

First, recall the *forgetful functor*,
recursively replace $t{:}X$ with $\Box X$.

This maps justification Z to modal Z°.

We build on justification logic J.

We add the axiom scheme

$$t{:}(u{:}X_1 \supset X_2) \supset g(t){:}X_3$$

where X_1, X_2, and X_3 are justification formulas such that $X_1^\circ = X_2^\circ = X_3^\circ$

Note that X_1, X_2, X_3 don't have to be the same!

A Realization Example

The GL theorem

$$\Box(\Box(X \vee Y) \supset (X \wedge Z)) \supset \Box X$$

is realized by

$$v_1{:}([v_2 + g(a)]{:}(X \vee Y) \supset (X \wedge Z)) \supset g(b){:}X$$

Where

$$v_4{:}X \vdash_{JGL} a{:}(v_5{:}(X \vee Y) \supset (X \vee Y))$$

and

$$v_1{:}[g(a){:}(X \vee Y) \supset (X \wedge Z)] \vdash_{JGL} b{:}(v_4{:}X \supset X)$$

(both from internalization)

A comment:

In

$$t{:}(u{:}X_1 \supset X_2) \supset g(t){:}X_3$$

it looks like $g(t)$ only depends on t,
and not on u.

Worked out examples show that t generally is built up from justification terms that include u.

I don't really understand this yet.

Second comment:

I don’t have a justification semantics for this logic.

Canonical machinery won’t work,
but I don't know what will.

Realization is proved constructively,
using a tableau system for GL.

Going Further

Similar ideas work
for Grzegorczyk logic.

Add to LP

$$s{:}(t{:}(X_1 \supset u{:}X_2) \supset X_3) \supset [g(s)]{:}X_4,$$
where $X_1^\circ = X_2^\circ = X_3^\circ = X_4^\circ$

Again, realization is constructive,
but a semantics is missing.

And for both *GL* and *Grz*,
what benefit does
a justification counterpart bring?

That's where things are at the moment.

Thank You